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BRST approach to Lagrangian formulation for mixed-symmetry fermionic higher-spin fields

Pavel Yu. Moshin[ab] **and Alexander A. Reshetnyak**[c]

[a] *Instituto de Física, Universidade de São Paulo,*
Rua do Matão, Travessa R, 187, Caixa Postal 66318-CEP, 05315-970 São Paulo, Brazil
[b] *Tomsk State Pedagogical University,*
Komsomolskiy Ave. 75, 634041 Tomsk, Russia
[c] *Laboratory of Non-equilibrium State Theory,*
Institute of Strength Physics and Materials Science,
Akademicheskiy Ave. 2/1, 634021 Tomsk, Russia
E-mail: moshin@dfn.if.usp.br, reshet@tspu.edu.ru

Abstract: We construct a Lagrangian description of irreducible half-integer higher-spin representations of the Poincare group with the corresponding Young tableaux having two rows, on a basis of the BRST approach. Starting with a description of fermionic higher-spin fields in a flat space of any dimension in terms of an auxiliary Fock space, we realize a conversion of the initial operator constraint system (constructed with respect to the relations extracting irreducible Poincare-group representations) into a first-class constraint system. For this purpose, we find auxiliary representations of the constraint subsuperalgebra containing the subsystem of second-class constraints in terms of Verma modules. We propose a universal procedure of constructing gauge-invariant Lagrangians with reducible gauge symmetries describing the dynamics of both massless and massive fermionic fields of any spin. No off-shell constraints for the fields and gauge parameters are used from the very beginning. It is shown that the space of BRST cohomologies with a vanishing ghost number is determined only by the constraints corresponding to an irreducible Poincare-group representation. To illustrate the general construction, we obtain a Lagrangian description of fermionic fields with generalized spin (3/2,1/2) and (3/2,3/2) on a flat background containing the complete set of auxiliary fields and gauge symmetries.

Keywords: Field Theories in Higher Dimensions, BRST Symmetry, Space-Time Symmetries, Gauge Symmetry.

The paper is dedicated to the memory of the sister of A.A.R., a mathematician and remarkable person, Olga Pavlicheva, nee Reshetnyak, who untimely passed away last year.

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Contents

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1. Introduction

The study of various aspects of higher-spin (HS) field theory has attracted a considerable attention for a long time due to the hope of discovering new possible approaches to the unification of the fundamental interactions. Higher-spin field theory is closely related to superstring theory, which operates with an infinite tower of bosonic and fermionic higher-spin fields. The problem of a covariant Lagrangian description of fields with an arbitrary spin propagating on flat [1 – 15] and (A)dS [16 – 27] backgrounds as well as the problem of constructing an interacting higher-spin field theory are in the permanent focus of research (for reviews and more references, see, e.g., [28]). One of the attractive features of investigating higher-spin gauge theories in AdS spaces is due to a possible relation of this study to the tensionless limit of superstring theory on the $AdS_5 \times S_5$ Ramond-Ramond background [29, 30] and the conformal $\mathcal{N} = 4$ SYM theory in the context of the AdS/CFT correspondence [31].

At present, the dynamics of totally symmetric higher-spin fields presents the most well-developed direction in the variety of unitary representations of the Poincare and AdS algebras [2, 3, 16, 17, 21]. To a great extent, this is caused by the fact that in a $4d$ space-time there is no place for mixed-symmetry irreducible representations with the exception of dual

theories.[1] In higher space-time dimensions, there appear mixed-symmetry representations determined by more than one spin-like parameters, and the problem of their field-theoretic description is not so well-developed as for totally symmetric irreps. Starting from the papers of Fierz-Pauli and Singh-Hagen [1, 2] for higher-spin field theories in the Minkowski space, it has been known that all such theories include, together with the basic fields of a given spin, also some auxiliary fields of lower spins, necessary to provide a compatibility of the Lagrangian equations of motion with the relations that determine irreducible representations of the Poincare group. Attempts to construct Lagrangian descriptions of free and interacting higher-spin field theories have resulted in consistency problems, which are not completely resolved until now.

The present work is devoted to the construction of gauge-invariant Lagrangians for both massless and massive mixed-symmetry spin-tensor fields of rank $n_1 + n_2 + \ldots + n_k$, with any integer numbers $n_1 \geq n_2 \geq \ldots \geq n_k \geq 1$ for $k = 2$ in a d-dimensional Minkowski space, the fields being elements of Poincare-group irreps with a Young tableaux having two rows. In the case of the Minkowski space, several approaches have been proposed to study mixed-symmetry higher-spin fields [7, 8, 11, 12]. Our approach is based on the BFV-BRST construction [33], see also the reviews [34, 35], which was initially developed for a Hamiltonian quantization of dynamical systems subject to first-class constraints. Following a tradition accepted in string theory and higher-spin field theory, we further refer to this method as the BRST method, and to the corresponding BFV charge, as the BRST operator. The application of the BRST construction to higher-spin field theory consists of three steps. First, the conditions that determine the representations with a given spin are regarded as a system of first- and second-class operator constraints in an auxiliary Fock space. Second, the system of the initial constraints is converted, with a preservation of the initial algebraic structure, into a system of first-class constraints alone in an enlarged Fock space (see [36] for the development of conversion methods), with respect to which one constructs the BRST charge. Third, the Lagrangian for a higher-spin field is constructed in terms of the BRST charge in such a way that the corresponding equations of motion reproduce the initial constraints. We emphasize that this approach automatically implies a gauge-invariant Lagrangian description reflecting the general fact of BV-BFV duality [37, 38], realized in order to reproduce a Lagrangian action or a probability amplitude by means of a Hamiltonian object.

The construction of the flat dynamics of mixed-symmetry gauge fields has been examined in [4, 5, 7 – 9, 11, 12], including the construction of Lagrangians in the BRST approach for massless bosonic higher-spin fields with two rows of the Young tableaux [9], and recently also for interacting bosonic HS fields [39] and for those of lower spins [40] on the basis of the BV cohomological deformation theory [41]. Lagrangian descriptions of massless mixed-symmetry fermionic and bosonic higher-spin fields in the (A)dS spaces have been suggested within a "frame-like" approach in [27], whereas for massive fields of lower superspins in the flat and (A)dS spaces they have been examined in [13]. To be complete, note that for free totally symmetric higher-spin fields of integer spins the BRST approach has been used to

[1] For a detailed discussion of dual theories in various dimensions, see [11, 28, 32].

derive Lagrangians in the flat space [8, 42] and in the (A)dS space [43]. The corresponding programme of a Lagrangian description of fermionic HS fields has been realized in the flat space [44] and in the (A)dS space [45].

In this paper, we construct a gauge-invariant Lagrangian description of fermionic HS fields in a Minkowski space of any dimension, corresponding to a unitary irreducible Poincare-group representation with the Young tableaux having two rows of length n_1, n_2 ($n_1 \geq n_2$).

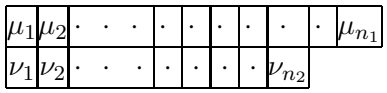
The paper is organized as follows. In section 2, we formulate a closed Lie superalgebra of operators, based on the constraints in an auxiliary Fock space that determines an irreducible representation of the Poincare group with a generalized spin $\mathbf{s} = (n_1+1/2, n_2+1/2)$. In section 3, we construct a Verma module, being an auxiliary representation for a rank-2 subsuperalgebra of the superalgebra of the initial constraints corresponding to the subsystem of second-class constraints. This representation is then realized in terms of new (additional) creation and annihilation operators in Fock space. Note that a similar construction for bosonic HS fields in a flat space has been presented in [46]. In section 4, we carry out a conversion of the initial system of first- and second-class constraints into a system of first-class constraints in the space being the tensor product of the initial and new Fock spaces. Next, we construct the BRST operator for the converted constraint superalgebra. The construction of an action and of a sequence of reducible gauge transformations describing the propagation of a mixed-symmetry fermionic field of an arbitrary spin is realized in section 5. We show that the Lagrangian description for a theory of a massive half-integer mixed-symmetry HS field in a d-dimensional Minkowski space is deduced by dimensional reduction of a massless HS field theory of the same type in a $(d+1)$-dimensional flat space. In section 6, we sketch a proof of the fact that the resulting action reproduces the correct conditions for a field that determine an irreducible representation of the Poincare group with a fixed $\mathbf{s} = (n_1 + 1/2, n_2 + 1/2)$ spin. We illustrate the general formalism by a construction of gauge-invariant Lagrangian actions for massless and massive spin-$(1 + 1/2, 1/2)$ and spin-$(1 + 1/2, 1 + 1/2)$ fields in section 7. In Conclusion, we summarize the results of this work and outline some open problems.

In addition to the conventions of [9, 44, 46], we use the notation $\varepsilon(A)$, $gh(A)$ for the respective values of Grassmann parity and ghost number of a quantity A, and denote by $[A, B\}$ the supercommutator of quantities A, B, which in the case of definite values of Grassmann parity is given by $[A\,, B\} = AB - (-1)^{\varepsilon(A)\varepsilon(B)}BA$.

2. Half-integer HS symmetry algebra in flat space-time

In general, a massless half-integer irreducible representation of the Poincare group in a d-dimensional Minkowski space is described by a spin-tensor field $\Phi_{\mu_1\ldots\mu_{n_1},\nu_1\ldots\nu_{n_2},\ldots,\rho_1\ldots\rho_{n_k}}(x)$, with the Dirac index being suppressed, of rank $n_1 + n_2 + \ldots + n_k$ and generalized spin $\mathbf{s} = (n_1 + 1/2, n_2 + 1/2, \ldots, n_k + 1/2)$, which corresponds to a Young tableaux with k rows of length $n_1, n_2, \ldots, n_k$, respectively, and $k \leq [(d-1)/2]$. This field is symmetric with respect to the permutations of each type of indices μ_i, $i = 1, \ldots, k$.

In this paper, we restrict ourselves to the fields characterized by a Young tableaux with $k = 2$ rows:



(2.1)

The field $\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) \equiv \Phi_{\mu_1\dots\mu_{n_1},\nu_1\dots\nu_{n_2}}(x)$, as an element of a Poincare-group irrep, obeys the mass-shell and γ-traceless conditions for each type of indices[2]

$$\imath\gamma^\mu\partial_\mu\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) = 0\,, \tag{2.2}$$

$$\gamma^{\mu_1}\Phi_{\mu_1\mu_2\dots\mu_{n_1},(\nu)_{n_2}}(x) = 0\,, \tag{2.3}$$

$$\gamma^{\nu_1}\Phi_{(\mu)_{n_1},\nu_1\nu_2\dots\nu_{n_2}}(x) = 0\,. \tag{2.4}$$

The correspondence with a given Young tableaux implies that after the symmetrization of all the vector indices of the first row with any vector index of the second row the field $\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x)$ becomes equal to zero:

$$\Phi_{\{(\mu)_{n_1},\nu_1\}\nu_2\dots\nu_{n_2}}(x) \equiv \sum_{i=1}^{n_1}\Phi_{\mu_1\dots\mu_{i-1}\nu_1\mu_{i+1}\dots\mu_{n_1},\mu_i\nu_2\dots\nu_{n_2}}(x) + \Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) = 0\,, \tag{2.5}$$

where in the case $i = 1$ it is implied that $\Phi_{\mu_0\nu_1\mu_2\dots\mu_{n_1},\mu_1\nu_2\dots\nu_{n_2}}(x) \equiv \Phi_{\nu_1\mu_2\dots\mu_{n_1},\mu_1\nu_2\dots\nu_{n_2}}(x)$.

In order to describe all the irreducible representations simultaneously, it is convenient to introduce an auxiliary Fock space $\mathcal{H}$ generated by creation and annihilation operators a^{i+}_μ, a^j_μ with additional internal indices, $i, j = 1, 2$,

$$[a^i_\mu, a^{j+}_\nu] = -\eta_{\mu\nu}\delta^{ij}\,, \qquad \delta^{ij} = diag(1, 1)\,. \tag{2.6}$$

The general state (a Dirac-like spinor) of the Fock space has the form

$$|\Phi\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n_1}\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x)\,a_1^{+\mu_1}\dots a_1^{+\mu_{n_1}}a_2^{+\nu_1}\dots a_2^{+\nu_{n_2}}|0\rangle, \tag{2.7}$$

providing the symmetry property of $\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x)$ under the permutation of indices of the same type. We refer to the vector (2.7) as the basic vector.

Because of the property of translational invariance of the vacuum, $\partial_\mu|0\rangle = 0$, the conditions (2.2)–(2.4) can be equivalently expressed in terms of the bosonic operators

$$\tilde{t}_0 = i\gamma^\mu\partial_\mu\ , \qquad \tilde{t}^i = \gamma^\mu a^i_\mu\ , \tag{2.8}$$

$$t\ = a^{1+}_\mu a^{2\mu} \tag{2.9}$$

as follows:

$$\tilde{t}_0|\Phi\rangle = \tilde{t}^i|\Phi\rangle = t|\Phi\rangle = 0. \tag{2.10}$$

Thus, the constraints (2.10) with each component $\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x)$ of the vector (2.7) subject to (2.2)–(2.4) describe a field of spin $(n_1 + 1/2, n_2 + 1/2)$.

[2]Throughout the paper, we use the mostly minus signature $\eta_{\mu\nu} = diag(+,-,\dots,-)$, $\mu,\nu = 0,1,\dots,d-1$, and the Dirac matrices satisfy the relations $\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}$.

Because of the fermionic nature of equations (2.2)–(2.4) with respect to the standard Lorentz-like Grassmann parity, and due to the bosonic nature of the primary constraint operators $\tilde{t}_0, \tilde{t}^i$, $\varepsilon(\tilde{t}_0) = \varepsilon(\tilde{t}^i) = 0$, in order to equivalently transform these operators into fermionic ones, we now introduce a set of $d+1$ Grassmann-odd gamma-matrix-like objects $\tilde{\gamma}^\mu$, $\tilde{\gamma}$, subject to the conditions

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}, \qquad \{\tilde{\gamma}^\mu, \tilde{\gamma}\} = 0, \qquad \tilde{\gamma}^2 = -1, \tag{2.11}$$

and related to the conventional gamma-matrices as follows:[3]

$$\gamma^\mu = \tilde{\gamma}^\mu \tilde{\gamma}. \tag{2.12}$$

We can now define Grassmann-odd constraints,

$$t_0 = -\imath\tilde{\gamma}^\mu \partial_\mu\,, \qquad t^i = \tilde{\gamma}^\mu a^i_\mu, \tag{2.13}$$

related to the operators (2.8) as follows:

$$\left(t_0, t^i\right) = \tilde{\gamma}\left(-\tilde{t}_0, \tilde{t}^i\right). \tag{2.14}$$

We next define an odd scalar product:

$$\langle\tilde{\Psi}|\Phi\rangle = \int d^d x \sum_{n_1,k_1,n_2,k_2=0}^{\infty} \langle 0|a_1^{\rho_1} \dots a_1^{\rho_{k_1}} a_2^{\sigma_1} \dots a_2^{\sigma_{k_2}} \Psi^+_{(\rho)_{k_1},(\sigma)_{k_2}}(x)\tilde{\gamma}_0 \Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x) \times$$
$$a_1^{+\mu_1} \dots a_1^{+\mu_{n_1}} a_2^{+\nu_1} \dots a_2^{+\nu_{n_2}}|0\rangle. \tag{2.15}$$

The operators t_0, t^i, t in (2.9) and (2.13), with $t^{i+} = \tilde{\gamma}^\mu a^{i+}_\mu$ and $t^+ = a^{2+}_\mu a^{1\mu}$ being Hermitian conjugate, respectively, to t^i, t with reference to the scalar product (2.15), generate an operator Lie superalgebra composed of the operators[4]

$$t_0 = -\imath\tilde{\gamma}^\mu\partial_\mu\,, \tag{2.16}$$
$$t^i = \tilde{\gamma}^\mu a^i_\mu\,, \qquad t^{i+} = \tilde{\gamma}^\mu a^{i+}_\mu\,, \tag{2.17}$$
$$t = a^{1+}_\mu a^{2\mu}\,, \qquad t^+ = a^{2+}_\mu a^{1\mu}\,, \tag{2.18}$$
$$l^i = -ia^i_\mu\partial^\mu\,, \qquad l^{i+} = -ia^{i+}_\mu\partial^\mu\,, \tag{2.19}$$
$$l^{ij} = \tfrac{1}{2}\,a^{i\mu}a^j_\mu\,, \qquad l^{ij+} = \tfrac{1}{2}\,a^{i\mu+}a^{j+}_\mu\,, \tag{2.20}$$
$$l_0 = \partial^\mu\partial_\mu\,, \qquad g^i_0 = -a^{i+}_\mu a^{i\mu} + \tfrac{d}{2}\,, \tag{2.21}$$

which is invariant under Hermitian conjugation.

The operators (2.16)–(2.21) form a superalgebra given by table 1, with an omission of the Poincare-group Casimir operator l_0 being the central charge of this algebra, where the

[3] For more details, see [44].

[4] For the operators l^{12}, l^{12+} in (2.20), we have used a definition slightly different from that of [46], where $(\hat{l}^{12}, \hat{l}^{12+}) = 2(l^{12}, l^{12+})$.

$[\downarrow, \rightarrow\}$	t_0	t^i	t^{i+}	t	t^+	l^i	l^{i+}	l^{ij}	l^{ij+}	g_0^i
t_0	$-2l_0$	$2l^i$	$2l^{i+}$	0	0	0	0	0	0	0
t^k	$2l^k$	$4l^{ki}$	A^{ki}	$-t^2\delta^{k1}$	$-t^1\delta^{k2}$	0	$-t_0\delta^{ki}$	0	$B^{k,ij}$	$t^i\delta^{ki}$
t^{k+}	$2l^{k+}$	A^{ik}	$4l^{ki+}$	$t^{1+}\delta^{k2}$	$t^{2+}\delta^{k1}$	$t_0\delta^{ki}$	0	$C^{k,ij}$	0	$-t^{i+}\delta^{ki}$
t	0	$t^2\delta^{i1}$	$-t^{1+}\delta^{i2}$	0	$g_0^1 - g_0^2$	$l^2\delta^{i1}$	$-l^{1+}\delta^{i2}$	D^{ij}	E^{ij}	F^i
t^+	0	$t^1\delta^{i2}$	$-t^{2+}\delta^{i1}$	$g_0^2 - g_0^1$	0	$l^1\delta^{i2}$	$-l^{2+}\delta^{i1}$	G^{ij}	H^{ij}	I^i
l^k	0	0	$-t_0\delta^{ik}$	$-l^2\delta^{k1}$	$-l^1\delta^{k2}$	0	$l_0\delta^{ik}$	0	$J^{k,ij}$	$l^i\delta^{ik}$
l^{k+}	0	$t_0\delta^{ik}$	0	$l^{1+}\delta^{k2}$	$l^{2+}\delta^{k1}$	$-l_0\delta^{ik}$	0	$K^{k,ij}$	0	$-l^{i+}\delta^{ik}$
l^{kl}	0	0	$-C^{i,kl}$	$-D^{kl}$	$-G^{kl}$	0	$-K^{i,kl}$	0	$L^{kl,ij}$	$l^{i\{k}\delta^{l\}i}$
l^{kl+}	0	$-B^{i,kl}$	0	$-E^{kl}$	$-H^{kl}$	$-J^{i,kl}$	0	$-L^{ij,kl}$	0	$-l^{i\{k+}\delta^{l\}i}$
g_0^k	0	$-t_k\delta^{ik}$	$t^{k+}\delta^{ik}$	$-F^k$	$-I^k$	$-l^k\delta^{ik}$	$l^{k+}\delta^{ik}$	$-l^{k\{i}\delta^{j\}k}$	$l^{k\{i+}\delta^{j\}k}$	0

Table 1: The superalgebra of the initial operators

quantities A^{ik}, $B^{k,ij}$, $C^{k,ij}$, D^{ij}, E^{ij}, F^i, I^i, G^{ij}, H^{ij}, $J^{k,ij}$, $K^{k,ij}$, $L^{kl,ij}$ are defined by the relations

$$A^{ik} = -2g_0^i\delta^{ik} + 2t\delta^{i2}\delta^{k1} + 2t^+\delta^{i1}\delta^{k2}\,, \tag{2.22}$$

$$B^{k,ij} = -\frac{1}{2}t^{\{i+}\delta^{j\}k}\,, \qquad C^{k,ij} = \frac{1}{2}t^{\{i}\delta^{j\}k}\,, \tag{2.23}$$

$$D^{ij} = l^{\{i2}\delta^{j\}1}\,, \qquad E^{ij} = -l^{1\{i+}\delta^{j\}2}\,, \tag{2.24}$$

$$F^i = t(\delta^{i2} - \delta^{i1})\,, \qquad I^i = t^+(\delta^{i1} - \delta^{i2})\,, \tag{2.25}$$

$$G^{ij} = l^{1\{i}\delta^{j\}2}\,, \qquad H^{ij} = -l^{\{i2+}\delta^{j\}1}\,, \tag{2.26}$$

$$J^{k,ij} = -\frac{1}{2}l^{\{i+}\delta^{j\}k}\,, \qquad K^{k,ij} = \frac{1}{2}l^{\{i}\delta^{j\}k}\,, \tag{2.27}$$

$$\begin{aligned} L^{kl,ij} = \frac{1}{4}\Big\{ & \delta^{ik}\delta^{lj}\Big[2g_0^k\delta^{kl} + g_0^k + g_0^l\Big] \\ & - \delta^{ik}\Big[t\Big(\delta^{l2}(\delta^{j1}+\delta^{k1}\delta^{kj})+\delta^{k2}\delta^{j1}\delta^{lk}\Big)+t^+\Big(\delta^{l1}(\delta^{j2}+\delta^{k2}\delta^{kj})+\delta^{k1}\delta^{j2}\delta^{lk}\Big)\Big] \\ & - \delta^{lj}\Big[t\Big(\delta^{k2}(\delta^{i1}+\delta^{l1}\delta^{li})+\delta^{l2}\delta^{i1}\delta^{kl}\Big)+t^+\Big(\delta^{k1}(\delta^{i2}+\delta^{l2}\delta^{li})+\delta^{l1}\delta^{i2}\delta^{lk}\Big)\Big]\Big\}. \end{aligned} \tag{2.28}$$

We call this algebra the half-integer higher-spin symmetry algebra in Minkowski space with a Young tableaux having two rows.

From the viewpoint of constraint system theory, the above superalgebra is a system of constraints, except for the operators g_0^k, being non-degenerate in $\mathcal{H}$. These operators,

as follows from table 1, determine an invertible operator supermatrix of commutators for the subsystem of second-class constraints, $\{t_k, t_k^+, l_{ij}, l_{ij}^+, t, t^+\}$, with the other constraints, t_0, l_0, l_i, l_i^+, being first-class ones.

A conversion of this constraint system $\{o_I\}$, including the operators g_0^k, into a first-class constraint system $\{O_I\}$ by means of an additive composition of o_I with some operators o'_I depending on new creation and annihilation operators, $o_I \to O_I = o_I + o'_I$, can be effectively realized only for the subsuperalgebra of the entire symmetry superalgebra that contains the subsystem of second class-constraints and g_0^k. The only requirement, as shown in [46], is that each of the Hermitian operators g_0^k should contain linearly an arbitrary parameter h^k, whose values are to be determined later.

3. Auxiliary representation for the superalgebra with second-class constraints

In this section, we describe the method of Verma module construction for the Lie superalgebra with second-class constraints alone. Having denoted $\{o_a\} = \{t_k, t_k^+, l_{ij}, l_{ij}^+, g_0^k\}$, $o_a \in \{o_I\}$, as the basis elements of the above superalgebra, and using the requirements that o_a, o'_a must supercommute, $\{o_a, o'_a] = 0$, and that the converted constraints must be in involution, $\{O_a, O_b] \sim O_c$, we find that the superalgebra of the additional parts o'_a is uniquely determined by the same algebraic conditions as those for the initial constraints. In this case, it is unnecessary to convert the subsystem of the initial first-class constraints not entering $\{o_a\}$, and therefore they remain intact.

Following [44] and the general method of Verma module construction for mixed-symmetry integer-spin HS fields [46], let us denote $E^\alpha \equiv (t^k; l^{ij}, t) = (E^{\alpha_0}; E^{\alpha_1})$, $(\alpha_0 > 0, \alpha_1 > 0)$ for $i \leq j$, and define

$$\mathcal{H}^i = g_0^i + g_0'^i\,, \quad g_0'^i = h^i + \dots\,, \qquad \mathcal{E}^\alpha = E^\alpha + E'^\alpha(h)\,, \quad \alpha_0 = 1,2\,,\ \alpha_1 = 1,2,3,4\,. \tag{3.1}$$

The quantities $g_0^i, E^{\alpha_1}, E^{-\alpha_1}$ are the Cartan generators, positive and negative root vectors, except for $\alpha_1 = 2$ (see footnote 4) of the subalgebra $so(3,2)$ in the superalgebra of second-class constraints, and the odd generators $E^{\alpha_0}, E^{-\alpha_0}$ supplement the basis $so(3,2)$ up to that of the above superalgebra. The quantities $g_0'^i, E'^\alpha, E^{-\prime\alpha}$ and $\mathcal{H}^i, \mathcal{E}^\alpha, \mathcal{E}^{-\alpha}$ have the same identification respectively for the additional and enlarged operators of the symmetry superalgebra.

Consider the highest-weight representation of the superalgebra of the additional parts with the highest-weight vector $|0\rangle_V$ annihilated by the positive roots and being the proper vector of the Cartan generators:

$$E'^\alpha|0\rangle_V = 0\,, \alpha > 0\,, \qquad g_0'^i|0\rangle_V = h^i|0\rangle_V\,. \tag{3.2}$$

Following the Poincare-Birkhoff-Witt theorem, the basis space of this representation, called in the mathematical literature the Verma module [47], is given by the vectors

$$\big|\vec{n}_k^0, \vec{n}_{ij}, n\rangle_V = \big(E'^{-\alpha_1^0}\big)^{n_1^0}\big(E'^{-\alpha_2^0}\big)^{n_2^0}\big(E'^{-\alpha_1^1}\big)^{n_{11}}\big(E'^{-\alpha_1^2}\big)^{n_{12}}\big(E'^{-\alpha_1^3}\big)^{n_{22}}\big(E'^{-\alpha_1^4}\big)^{n}|0\rangle_V\,, \tag{3.3}$$

where $\vec{n}^0_k = (n^0_1, n^0_2)$, $\vec{n}_{ij} = (n_{11}, n_{12}, n_{22})$, $n^0_1, n^0_2 = 0, 1$, $n_{ij} \in \mathbb{N}_0$. Note that the restriction for the values of $\vec{n}^0_k$ in (3.3) is due to the identities

$$\{E'^{-\alpha^0_i}, E'^{-\alpha^0_j}] = 4(E'^{-\alpha^1_1}, E'^{-\alpha^2_1}, E'^{-\alpha^3_1}) = 4(l'^+_{11}, l'^+_{12}, l'^+_{22}), \qquad i, j = 1, 2, i \leq j\,. \tag{3.4}$$

Using the commutation relations of the superalgebra given by table 1 and the formula for the product of graded operators,

$$\begin{aligned} AB^n &= \sum_{k=0}^{n} (-1)^{\varepsilon(A)\varepsilon(B)(n-k)} C^{(s)n}_{\ k} B^{n-k} \mathrm{ad}^k_B A\,, \quad n \geq 0\,, \quad s = \varepsilon(B)\,, \\ \mathrm{ad}^k_B A &= \mathrm{ad}_B \left(\mathrm{ad}^{k-1}_B A\right), \qquad\qquad \mathrm{ad}_B A = \{A, B\}\,, \end{aligned} \tag{3.5}$$

we can calculate the explicit form of the Verma module. In (3.5), we have introduced generalized coefficients for a number of graded combinations, $C^{(s)n}_{\ k}$, that coincide with the standard ones only for the bosonic operator B: $C^{(0)n}_{\ k} = C^n_k = \frac{n!}{k!(n-k)!}$. These coefficients are defined recursively, by the relations

$$C^{(s)n+1}_{\ k} = (-1)^{s(n+k+1)} C^{(s)n}_{\ k-1} + C^{(s)n}_{\ k}\,, \qquad n, k \geq 0\,, \tag{3.6}$$

$$C^{(s)n}_{\ 0} = C^{(s)n}_{\ n} = 1\,, \qquad C^{(s)n}_{\ k} = 0\,, \quad n < k \tag{3.7}$$

and possess the properties $C^{(s)n}_{\ k} = C^{(s)n}_{\ n-k}$. The corresponding values of $C^{(1)n}_{\ k}$ are defined, for $n \geq k$, by the formulae

$$C^{(1)n}_{\ k} = \sum_{i_k=1}^{n-k+1} \sum_{i_{k-1}=1}^{n-i_k-k+2} \cdots \sum_{i_2=1}^{n-\sum_{j=3}^{k} i_j - 1} \sum_{i_1=1}^{n-\sum_{j=2}^{k} i_j} (-1)^{k(n+1)+\sum_{j=1}^{[(k+1)/2]} (i_{2j-1}+1)}\,, \tag{3.8}$$

which follow by induction, and in which $[a]$ stands for the integer part of the number a. For our purposes, due to $n^0_k = 0, 1$ in (3.3), (3.5), it is sufficient to know that $C^{(1)0}_{\ 0} = C^{(1)1}_{\ 0} = 1$ and $C^{(1)n^0_k}_{\ 1} = n^0_k$.

Then, following [48] and making use of the mapping

$$\left|\vec{n}^0_k, \vec{n}_{ij}, n\right\rangle_V \leftrightarrow \left|\vec{n}^0_k, \vec{n}_{ij}, n\right\rangle = \left(f^+_1\right)^{n^0_1} \left(f^+_2\right)^{n^0_2} \left(b^+_{11}\right)^{n_{11}} \left(b^+_{12}\right)^{n_{12}} \left(b^+_{22}\right)^{n_{22}} \left(b^+\right)^n |0\rangle\,, \tag{3.9}$$

where $\left|\vec{n}^0_k, \vec{n}_{ij}, n\right\rangle$, for $n^0_k = 0, 1$, $n_{ij} \in \mathbb{N}_0$, are the basis vectors of a Fock space $\mathcal{H}'$ generated by new fermionic, f^+_k, f_k, $k = 0, 1$, and bosonic, b^+_{ij}, b^+, b_{ij}, b, $i, j = 1, 2, i \leq j$, creation and annihilation operators with the standard (only nonvanishing) commutation relations

$$\{f_k\,, f^+_l\} = \delta_{kl}\,, \qquad [b_{ij}, b^+_{lk}] = \delta_{il}\delta_{jk}\,,\ i \leq j, k \leq l\,, \qquad [b\,, b^+] = 1\,, \tag{3.10}$$

we can represent the Verma module as polynomials in the creation operators of the Fock space $\mathcal{H}'$.

First, we find the action of the negative root operators $E'^{-\alpha}$ on the basis vectors. After a simple calculation, one obtains

$$
\begin{aligned}
t'^{i+}\big|\vec{n}_k^0,\vec{n}_{lm},n\rangle_V=&\delta_{i1}\left(1+\left[\frac{n_1^0+1}{2}\right]\right)\left|n_1^0+1 mod 2,n_2^0,n_{11}+\left[\frac{n_1^0+1}{2}\right],n_{12},n_{22},n\right\rangle_V \qquad (3.11)\\
&+\delta_{i2}(-1)^{n_1^0}\Bigg\{\left(1+\left[\frac{n_2^0+1}{2}\right]\right)\left|n_1^0,n_2^0+1 mod 2,n_{11},n_{12},n_{22}+\left[\frac{n_2^0+1}{2}\right],n\right\rangle_V\\
&-4n_1^0\left|n_1^0-1,n_2^0,n_{11},n_{12}+1,n_{22},n\right\rangle_V\Bigg\},\\
l'^{+}_{ij}\left|\vec{n}_k^0,\vec{n}_{lm},n\right\rangle_V &= \left|\vec{n}_k^0,\vec{n}_{lm}+\delta_{il}\delta_{jm},n\right\rangle_V\ , \qquad (3.12)\\
t'^{+}\left|\vec{n}_k^0,\vec{n}_{lm},n\right\rangle_V &= \left|\vec{n}_k^0,\vec{n}_{lm},n+1\right\rangle_V-2n_{11}\left|\vec{n}_k^0,n_{11}-1,n_{12}+1,n_{22},n\right\rangle_V \qquad (3.13)\\
&\quad -n_1^0\left(1+\left[\frac{n_2^0+1}{2}\right]\right)\left|n_1^0-1,n_2^0+1 mod 2,n_{11},n_{12},n_{22}+\left[\frac{n_2^0+1}{2}\right],n\right\rangle_V\\
&\quad -n_{12}\left|\vec{n}_k^0,n_{11},n_{12}-1,n_{22}+1,n\right\rangle_V\ ,\\
g_0'^{i}\left|\vec{n}_k^0,\vec{n}_{lm},n\right\rangle_V &= \left(n_k^0\delta_{ik}+\sum_{l\le m}n_{lm}\left(\delta^{il}+\delta^{im}\right)+n\left(\delta^{i2}-\delta^{i1}\right)+h^i\right)\left|\vec{n}_k^0,\vec{n}_{lm},n\right\rangle_V\ . \qquad (3.14)
\end{aligned}
$$

Second, for the positive root operators E'^{α} we find

$$
\begin{aligned}
t'^{1}\big|\vec{n}_k^0,\vec{n}_{lm},n\rangle_V=&-2n_1^0(2n_{11}+n_{12}-n+h^1)\left|n_1^0-1,n_2^0,\vec{n}_{lm},n\right\rangle_V \qquad (3.15)\\
&-(-1)^{n_1^0}\left(1+\left[\frac{n_1^0+1}{2}\right]\right)n_{11}\left|n_1^0+1 mod 2,n_2^0,n_{11}-1+\left[\frac{n_1^0+1}{2}\right],n_{12},n_{22},n\right\rangle_V\\
&-(-1)^{n_1^0+n_2^0}\frac{n_{12}}{2}\left(1+\left[\frac{n_2^0+1}{2}\right]\right)\left|n_1^0,n_2^0+1 mod 2,n_{11},n_{12}-1,n_{22}+\left[\frac{n_2^0+1}{2}\right],n\right\rangle_V\\
&+2(-1)^{n_1^0}n_2^0\Big(-n_{12}\left|n_1^0,n_2^0-1,n_{11},n_{12}-1,n_{22}+1,n\right\rangle_V\\
&+\left|n_1^0,n_2^0-1,\vec{n}_{lm},n+1\right\rangle_V\Big),\\
t'^{2}\left|\vec{n}_k^0,\vec{n}_{lm},n\right\rangle_V=&(-1)^{n_1^0}\Big\{-2n_2^0(2n_{22}-n_1^0+n+h^2)\left|n_1^0,n_2^0-1,\vec{n}_{lm},n\right\rangle_V \qquad (3.16)\\
&-\frac{n_{12}}{2}\left(1+\left[\frac{n_1^0+1}{2}\right]\right)\left|n_1^0+1 mod 2,n_2^0,n_{11}+\left[\frac{n_1^0+1}{2}\right],n_{12}-1,n_{22},n\right\rangle_V\\
&-(-1)^{n_2^0}n_{22}\left(1+\left[\frac{n_2^0+1}{2}\right]\right)\left|n_1^0,n_2^0+1 mod 2,n_{11},n_{12},n_{22}-1+\left[\frac{n_2^0+1}{2}\right],n\right\rangle_V\\
&-2n_1^0\Big(n(h^1-h^2-n+1)\left|n_1^0-1,n_2^0,\vec{n}_{lm},n-1\right\rangle_V\\
&-2n_{22}\left|n_1^0-1,n_2^0,n_{11},n_{12}+1,n_{22}-1,n\right\rangle_V\\
&-n_{12}\left|n_1^0-1,n_2^0,n_{11}+1,n_{12}-1,n_{22},n\right\rangle_V\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
l'^{11}\,\big|\vec{n}_k^0,\vec{n}_{lm},n\rangle_V \;&=\; n_{11}(n_{11}{+}n_{12}{+}n_1^0{-}n{-}1{+}h^1)\big|\vec{n}_k^0,n_{11}{-}1,n_{12},n_{22},n\rangle_V \\
&-\frac{n_{12}}{2}\big|\vec{n}_k^0,n_{11},n_{12}{-}1,n_{22},n{+}1\rangle_V{+}\frac{n_{12}(n_{12}{-}1)}{4}\big|\vec{n}_k^0,n_{11},n_{12}{-}2,n_{22}{+}1,n\rangle_V \\
&-n_1^0\Big\{2n_2^0\Big(\Big|\overrightarrow{n_k^0{-}1},\vec{n}_{lm},n+1\rangle_V-n_{12}\Big|\overrightarrow{n_k^0{-}1},n_{11},n_{12}-1,n_{22}+1,n\rangle_V\Big) \\
&-\frac{(-1)^{n_2^0}}{2}n_{12}\left(1{+}\left[\frac{n_2^0{+}1}{2}\right]\right)\Big|n_1^0{-}1,n_2^0{+}1mod2,n_{11},n_{12}{-}1,n_{22}{+}\left[\frac{n_2^0{+}1}{2}\right],n\Big\rangle_V\Big\},
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
l'^{12}\,\big|\vec{n}_k^0,\vec{n}_{lm},n\rangle_V \;&=\; \frac{n_{12}}{4}\left(2n_{11}{+}n_{12}{+}2n_{22}{+}\sum_k(n_k^0+h^k){-}1\right)\big|\vec{n}_k^0,n_{11},n_{12}-1,n_{22},n\rangle_V \\
&+\frac{1}{2}nn_{11}(h^2-h^1+n-1)\,\big|\vec{n}_k^0,n_{11}-1,n_{12},n_{22},n-1\rangle_V \\
&+n_{11}n_{22}\,\big|\vec{n}_k^0,n_{11}-1,n_{12}+1,n_{22}-1,n\rangle_V \\
&-\frac{n_{22}}{2}\,\big|\vec{n}_k^0,n_{11},n_{12},n_{22}-1,n+1\rangle_V \\
&+\frac{n_1^0}{2}\Big\{2n_2^0(n{+}2n_{22}{+}h^2)\,\Big|\overrightarrow{n_k^0{-}1},\vec{n}_{lm},n\rangle_V+(-1)^{n_2^0}\left(1{+}\left[\frac{n_2^0{+}1}{2}\right]\right)\times \\
&\qquad\times n_{22}\Big|n_1^0{-}1,n_2^0{+}1mod2,n_{11},n_{12},n_{22}{-}1{+}\left[\frac{n_2^0{+}1}{2}\right],n\Big\rangle_V\Big\} \\
&+\frac{n_2^0}{2}n_{11}\left(1{+}\left[\frac{n_1^0{+}1}{2}\right]\right)\Big|n_1^0{+}1mod2,n_2^0{-}1,n_{11}{-}1{+}\left[\frac{n_1^0{+}1}{2}\right],n_{12},n_{22},n\Big\rangle_V,
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
l'^{22}\,\big|\vec{n}_k^0,\vec{n}_{lm},n\rangle_V \;&=\; n_{22}(n_{12}{+}n{+}n_2^0{+}n_{22}{-}1{+}h^2)\,\big|\vec{n}_k^0,n_{11},n_{12},n_{22}{-}1,n\rangle_V \\
&+\frac{n_{12}n}{2}(n-1+h^2-h^1)\,\big|\vec{n}_k^0,n_{11},n_{12}-1,n_{22},n-1\rangle_V\,, \\
&+\frac{n_{12}(n_{12}-1)}{4}\,\big|\vec{n}_k^0,n_{11}+1,n_{12}-2,n_{22},n\rangle_V \\
&+\frac{n_2^0n_{12}}{2}\left(1{+}\left[\frac{n_1^0{+}1}{2}\right]\right)\Big|n_1^0{+}1mod2,n_2^0-1,n_{11}{+}\left[\frac{n_1^0{+}1}{2}\right],n_{12}{-}1,n_{22},n\Big\rangle_V,
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
t'\,\big|\vec{n}_k^0,\vec{n}_{lm},n\rangle_V \;&=\; n(h^1-h^2-n+1)\,\big|\vec{n}_k^0,\vec{n}_{lm},n-1\rangle_V \\
&-n_{12}\,\big|\vec{n}_k^0,n_{11}{+}1,n_{12}{-}1,n_{22},n\rangle_V-2n_{22}\,\big|\vec{n}_k^0,n_{11},n_{12}{+}1,n_{22}{-}1,n\rangle_V \\
&-n_2^0\left(1{+}\left[\frac{n_1^0{+}1}{2}\right]\right)\Big|n_1^0{+}1mod2,n_2^0{-}1,n_{11}{+}\left[\frac{n_1^0{+}1}{2}\right],n_{12},n_{22},n\Big\rangle_V.
\end{aligned}
\tag{3.20}
$$

Using expressions (3.11)–(3.20) and the mapping (3.9), we reconstruct the action of the operators E'^{α}, $E'^{-\alpha}$, $g_0'^{i}$ in the Fock space $\mathcal{H}'$, namely,

$$
t'^{i+} \;=\; f_i^+ + 2b_{ii}^+f_i + 4\delta_{i2}b_{12}^+f_1\,, \tag{3.21}
$$

$$
t'^{+} \;=\; b^+ - 2b_{12}^+b_{11} - b_{22}^+b_{12} - f_2^+f_1 + 2b_{22}^+f_1f_2\,, \tag{3.22}
$$

$$
g_0'^{i} \;=\; f_i^+f_i + \sum_{l\le m} b_{lm}^+b_{lm}(\delta^{il}+\delta^{im}) + b^+b(\delta^{i2}-\delta^{i1}) + h^i\,, \tag{3.23}
$$

$$
l'^{+}_{ij} \;=\; b_{ij}^+\,. \tag{3.24}
$$

$$
t'^{1} \;=\; -f_1^+b_{11} - \frac{1}{2}f_2^+b_{12} - 2\left(b_{11}^+b_{11}+b_{12}^+b_{12}-b^+b+h^1\right)f_1 + \left(2b^+-b_{22}^+b_{12}\right)f_2\,, \tag{3.25}
$$

$$t'^2 = -f_2^+ b_{22} - \frac{1}{2} f_1^+ b_{12} - 2\left(b_{22}^+ b_{22} - f_1^+ f_1 + b^+ b + h^2\right) f_2 \tag{3.26}$$
$$+2\left(h^1 - h^2 - b^+ b\right) b f_1 - \left(b_{11}^+ b_{12} + 4 b_{12}^+ b_{22}\right) f_1 \,,$$
$$l'^{11} = \left(b_{11}^+ b_{11} + b_{12}^+ b_{12} - b^+ b + f_1^+ f_1 + h^1\right) b_{11} - \frac{1}{2} b^+ b_{12} + \frac{1}{4} b_{22}^+ b_{12}^2 \tag{3.27}$$
$$+\frac{1}{2} f_2^+ f_1 b_{12} - 2 b^+ f_2 f_1 \,,$$

$$l'^{12} = \frac{1}{4}\Big[\sum_i \left(2 b_{ii}^+ b_{ii} + f_i^+ f_i + h^i\right) + b_{12}^+ b_{12}\Big] b_{12} + \frac{1}{2}\left(b^+ b + h^2 - h^1\right) b b_{11} \tag{3.28}$$
$$+\left(b_{12}^+ b_{12} - \frac{1}{2} b^+\right) b_{22} + \frac{1}{2} f_2^+ f_1 b_{22} + \frac{1}{2} f_1^+ f_2 b_{11}$$
$$+\left(b^+ b + b_{11}^+ b_{11} + b_{22}^+ b_{22} + h^2\right) f_2 f_1 \,,$$
$$l'^{22} = \left(b_{22}^+ b_{22} + b_{12}^+ b_{12} + b^+ b + f_2^+ f_2 + h^2\right) b_{22} + \frac{1}{2}\left(b^+ b + h^2 - h^1\right) b_{12} b \tag{3.29}$$
$$+\frac{1}{4} b_{11}^+ b_{12}^2 + \frac{1}{2} f_1^+ f_2 b_{12} + b_{11}^+ b_{12} f_2 f_1 \,,$$
$$t' = \left(h^1 - h^2 - b^+ b\right) b - b_{11}^+ b_{12} - 2 b_{12}^+ b_{22} - f_1^+ f_2 - 2 b_{11}^+ f_2 f_1 \,. \tag{3.30}$$

Note that the additional parts $E'^{\alpha}, E^{-\prime\alpha}$ do not obey the usual properties

$$\left(E'^{\alpha}\right)^+ \neq E^{-\prime\alpha} \,, \tag{3.31}$$

if one should use the standard rules of Hermitian conjugation for the new creation and annihilation operators,

$$(b_{ij})^+ = b_{ij}^+, \qquad (b)^+ = b^+, \qquad (f_i)^+ = f_i^+. \tag{3.32}$$

To restore the proper Hermitian conjugation properties for the additional parts, we change the scalar product in the Fock space $\mathcal{H}'$ as follows:

$$\langle \tilde{\Psi}_1 | \Psi_2 \rangle_{\rm new} = \langle \tilde{\Psi}_1 | K' | \Psi_2 \rangle \,, \tag{3.33}$$

for any vectors $|\Psi_1\rangle, |\Psi_2\rangle$ with some, yet unknown, operator K'. This operator is determined by the condition that all the operators of the algebra must have the proper Hermitian properties with respect to the new scalar product:

$$\langle \tilde{\Psi}_1 | K' E^{-\prime\alpha} | \Psi_2 \rangle = \langle \tilde{\Psi}_2 | K' E'^{\alpha} | \Psi_1 \rangle^*, \qquad \langle \tilde{\Psi}_1 | K' g_0'^i | \Psi_2 \rangle = \langle \tilde{\Psi}_2 | K' g_0'^i | \Psi_1 \rangle^*. \tag{3.34}$$

These relations permit one to determine the operator K', Hermitian with respect to the usual scalar product $\langle\,|\,\rangle$, as follows:

$$K' = Z^+ Z, \quad Z = \sum_{(\vec{n}_{lm}, n) \neq (\vec{0}, 0)}^{\infty} \sum_{\vec{n}_k^0 = (0,0)}^{(1,1)} \left|\vec{n}_k^0, \vec{n}_{lm}, n\right\rangle_V \frac{1}{(\vec{n}_{lm})! n!} \langle 0 | b^n b_{11}^{n_{11}} b_{12}^{n_{12}} b_{22}^{n_{22}} f_1^{n_1^0} f_2^{n_2^0}, \tag{3.35}$$

where $(\vec{n}_{lm})! = n_{11}! n_{12}! n_{22}!$. One can show by direct calculation that the following relation holds true: ${}_V\left\langle \vec{n}_k'^0, \vec{n}_{lm}', n' \,\middle|\, \vec{n}_k^0, \vec{n}_{lm}, n \right\rangle_V \sim \delta^{n_1^0 + 2n_{11} + n_{12} - n}_{n_1'^0 + 2n_{11}' + n_{12}' - n'} \delta^{n_2^0 + n_{12} + 2n_{22} + n}_{n_2'^0 + n_{12}' + 2n_{22}' + n'}$. For low pairs

of numbers $(n_1^0 + 2n_{11} + n_{12} - n, n_2^0 + n_{12} + 2n_{22} + n)$, with n, n_{ij} being the numbers of "particles" associated with b^+, b_{ij}^+ for $i \leq j$ (where b^+ reduces the spin number s_1 by one unit and increases the spin number s_2 by one unit simultaneously) and n_k^0 being the number of "particles" associated with f_k^+, the operator K' reads

$$\begin{aligned} K' = {} & |0\rangle\langle 0| + (h^1 - h^2)b^+|0\rangle\langle 0|b - 2h^i f_i^+|0\rangle\langle 0|f_i + 2f_2^+|0\rangle\langle 0|(h^1 - h^2)bf_1 \\ & + f_1^+ b^+|0\rangle\langle 0|\Big(2bf_1(h^2 - h^1)(h^1 - 1) + 2f_2(h^1 - h^2)\Big) \\ & + b_{12}^+|0\rangle\langle 0|\Big(\frac{1}{4}b_{12}(h^1 + h^2) + f_2 f_1 h^2 + \frac{1}{2}b_{11}b(h^2 - h^1)\Big) + h^i b_{ii}^+|0\rangle\langle 0|b_{ii} \\ & + b_{11}^+ b^+|0\rangle\langle 0|\Big(b_{11}b(h^1 - h^2)(h^1 - 1) + \frac{1}{2}b_{12}(h^2 - h^1) + 2f_2 f_1(h^2 - h^1)\Big) \\ & + f_1^+ f_2^+|0\rangle\langle 0|\Big(4f_2 f_1(h^2 h^1 + h^2 - h^1) + b_{12}h^2 + 2b_{11}b(h^2 - h^1)\Big) + \dots . \end{aligned} \tag{3.36}$$

This expression for the operator K' will be used later in constructing the examples of section 7.

Thus, we have constructed the additional parts o'_a, (3.21)–(3.30), for the constraints o_a. In the next section, we determine the algebra of the extended constraints and find the BRST operator corresponding to this algebra.

4. The converted superalgebra and the BFV-BRST operator

The superalgebra of the converted operators O_I,

$$O_I = (O_a, O_p), \qquad O_a = o_a + o'_a, \qquad O_p = o_p\,,\ o_p \in \{t_0, l_0, l_i, l_i^+\}\,, \tag{4.1}$$

has the same form as the superalgebra of the initial operators o_I, and therefore it is determined by the relations of table 1 under the replacement $o_I \leftrightarrow O_I$. Despite the fact that the operators $\mathcal{H}^i$ do not belong to the constraint system, and in order to provide a Lorentz-covariant description of BRST cohomology spaces, we do not impose the restrictions $\mathcal{H}^i|\chi\rangle_{\rm def} = 0$ on the vector $|\chi\rangle_{\rm def}$, being the vector $|\Phi\rangle$ (2.7) enlarged into the tensor product of the Fock spaces $\mathcal{H}_{\rm def} = \mathcal{H} \otimes \mathcal{H}'$,

$$|\chi\rangle_{\rm def} = \sum_{k_a} \big(f_i^+\big)^{k_i} \big(b_{lm}^+\big)^{k_{lm}} \big(b^+\big)^k\, a_1^{+\mu_1} \dots\, a_1^{+\mu_{k_{10}}} a_2^{+\nu_1} \dots\, a_2^{+\nu_{k_{20}}} \chi^{k_1 k_2 k_{11} k_{12} k_{22} k}_{(\mu)_{k_{10}},(\nu)_{k_{20}}}(x)\,|0\rangle, \tag{4.2}$$

and include $\mathcal{H}^i$ into the converted first-class constraint system, with respect to which we construct the BRST operator Q'. The sum in (4.2) is taken over k_{i0}, k_{lm}, k, running from 0 to infinity, and over k_i, running from 0 to 1 for $i = 0, 1$, $l, m = 1, 2$, $l \leq m$. Having constructed Q', we extract from it the operators $\mathcal{H}^i$, enlarged by means of the ghost variables $\mathcal{C}, \mathcal{P}$ up to new operators σ^i, $\sigma^i = \big(\mathcal{H}^i - h^i + O\,(\mathcal{C}\mathcal{P})\big)$, which will be used to describe, by virtue of the equations $(\sigma^i + h^i)|\chi\rangle = 0$, the direct sum of the Fock subspaces $\mathcal{H}_{(n_1,n_2)}$ of a definite generalized spin $\mathbf{s} = (n_1 + \frac{1}{2}, n_2 + \frac{1}{2})$ in the enlarged Hilbert space $\mathcal{H}_{\rm tot} = \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H}_{gh}$ for $|\chi\rangle \in \mathcal{H}_{\rm tot}$. In this case, the remaining operator Q, independent of the ghost variables $\eta^i_{\mathcal{H}}, \mathcal{P}^i_{\mathcal{H}}$ associated with $\mathcal{H}^i$, in $Q' = Q + O\left(\eta^i_{\mathcal{H}}, \mathcal{P}^i_{\mathcal{H}}\right)$, is covariant and nilpotent in each space $\mathcal{H}_{(n_1,n_2)}$ for the converted constraint system O_I without $\mathcal{H}^i$. Then,

substituting instead of the parameters $-h^i$ the operators σ^i, we obtain a nilpotent BRST operator in the complete space $\mathcal{H}_{\rm tot}$ without $\eta^i_{\mathcal{H}}, \mathcal{P}^i_{\mathcal{H}}$, which encodes the superalgebra of the converted constraints $\{O_I\} \setminus \{\mathcal{H}^i\}$ for fermionic HS fields with two rows of the Young tableaux.

The construction of a nilpotent fermionic BRST operator for a Lie superalgebra is based on a principle similar to those developed in [44, 45]: see the general analysis of the BFV quantization in the reviews [33 – 35]. Following the prescription of [34], the BRST operator constructed on a basis of the superalgebra presented in table 1 can be found in an exact form, with the use of the $(\mathcal{CP})$-ordering of the ghost coordinate $\mathcal{C}^I$ and momenta $\mathcal{P}_I$ operators, as follows:

$$Q' = O_I \mathcal{C}^I + \frac{1}{2} \mathcal{C}^I \mathcal{C}^J f^K_{JI} \mathcal{P}_K (-1)^{\varepsilon(O_K)+\varepsilon(O_J)} \tag{4.3}$$

with the constants f^K_{IJ} written in a compact x-local representation, $\{O_I\,, O_J] = f^K_{IJ} O_K$, and, according to table 1, Q' has the form

$$\begin{aligned}
Q' = {} & q_0 T_0 + q_i^+ T^i + T_i^+ q^i + \eta_0 L_0 + \eta_i^+ L^i + L_i^+ \eta^i + \eta^+_{lm} L^{lm} + L^+_{lm} \eta^{lm} + \eta^+ T + T^+ \eta + \eta^i_{\mathcal{H}} \mathcal{H}_i \\
& + \imath(\eta_i^+ q^i - \eta^i q_i^+) p_0 + (\eta^i_{\mathcal{H}} q_i + \eta_{ii} q_i^+) p_i^+ + (\eta^i_{\mathcal{H}} q_i^+ + \eta^+_{ii} q^i) p_i \\
& - \imath(q_0^2 - \eta_i^+ \eta^i) \mathcal{P}_0 - \imath(2 q^i q_i^+ - \eta^+_{ii} \eta^{ii}) \mathcal{P}^i_{\mathcal{H}} + (\eta^i_{\mathcal{H}} \eta_i^+ + \eta^+_{ii} \eta^i - 2 q_0 q_i^+) \mathcal{P}^i \\
& + (\eta_i \eta^i_{\mathcal{H}} + \eta_i^+ \eta^{ii} - 2 q_0 q^i) \mathcal{P}_i^+ + 2(\eta^i_{\mathcal{H}} \eta^+_{ii} - q_i^{+2}) \mathcal{P}_{ii} + 2(\eta_{ii} \eta^i_{\mathcal{H}} - q_i^2) \mathcal{P}^+_{ii} \\
& - 2 \left[\frac{1}{2} (\eta^1_{\mathcal{H}} + \eta^2_{\mathcal{H}}) \eta_{12} - \eta^+ \eta_{22} - \eta \eta_{11} + 2 q_1 q_2 \right] \mathcal{P}^+_{12} + \eta^+ \eta_{12} \mathcal{P}^+_{11} + \eta \eta_{12} \mathcal{P}^+_{22} \\
& + 2 \left[\frac{1}{2} (\eta^1_{\mathcal{H}} + \eta^2_{\mathcal{H}}) \eta^+_{12} - \eta^+ \eta^+_{11} - \eta \eta^+_{22} - 2 q_1^+ q_2^+ \right] \mathcal{P}_{12} - \eta \eta^+_{12} \mathcal{P}_{11} - \eta^+ \eta^+_{12} \mathcal{P}_{22} \\
& + \imath \eta \eta^+ \left(\mathcal{P}^1_{\mathcal{H}} - \mathcal{P}^2_{\mathcal{H}} \right) + \frac{\imath}{4} \eta^+_{12} \eta_{12} \left(\mathcal{P}^1_{\mathcal{H}} + \mathcal{P}^2_{\mathcal{H}} \right) \\
& + \left[\frac{1}{2} \eta^+_{12} \eta_{11} + \frac{1}{2} \eta^+_{22} \eta_{12} - 2 q_1 q_2^+ + (\eta^2_{\mathcal{H}} - \eta^1_{\mathcal{H}}) \eta^+ \right] \mathcal{P} \\
& + \left[\frac{1}{2} \eta^+_{11} \eta_{12} + \frac{1}{2} \eta^+_{12} \eta_{22} - 2 q_2 q_1^+ - (\eta^2_{\mathcal{H}} - \eta^1_{\mathcal{H}}) \eta \right] \mathcal{P}^+ \\
& + \left[\frac{1}{2} q_1 \eta^+_{12} - \eta^+ q_1^+ \right] p_2 + \left[\frac{1}{2} q_1^+ \eta_{12} - \eta q_1 \right] p_2^+ + \left[\frac{1}{2} q_2 \eta^+_{12} - \eta q_2^+ \right] p_1 \\
& + \left[\frac{1}{2} q_2^+ \eta_{12} - \eta^+ q_2 \right] p_1^+ + \left[\frac{1}{2} \eta^+_{12} \eta_2 - \eta \eta_2^+ \right] \mathcal{P}_1 + \left[\frac{1}{2} \eta_2^+ \eta_{12} + \eta^+ \eta_2 \right] \mathcal{P}_1^+ \\
& + \left[\frac{1}{2} \eta^+_{12} \eta_1 - \eta^+ \eta_1^+ \right] \mathcal{P}_2 + \left[\frac{1}{2} \eta_1^+ \eta_{12} - \eta_1 \eta \right] \mathcal{P}_2^+ .
\end{aligned} \tag{4.4}$$

Here, we imply summation over the repeated index i, and the raising (lowering) of the indices i, j in quantities f^{ij} is made by the two-dimensional Euclidian metric tensor g^{ij} (g_{ij}), $g^{ij} = diag(1, 1)$. The quantities q_0, q_i, q_i^+ and η_0, η_i^+, η_i, η^+_{lm}, η_{lm}, η, η^+, $\eta^i_{\mathcal{H}}$ are, respectively, bosonic and fermionic ghost "coordinates" corresponding to their canonically conjugate ghost "momenta" p_0, p_i^+, p_i, $\mathcal{P}_0$, $\mathcal{P}_i$, $\mathcal{P}_i^+$, $\mathcal{P}_{lm}$, $\mathcal{P}^+_{lm}$, $\mathcal{P}^+$, $\mathcal{P}$, $\mathcal{P}^i_{\mathcal{H}}$ for $i, l, m = 1, 2$, $l \leq m$. They form a set of Wick ghost pairs, $\left\{ \left(q_i, p_i^+\right), \left(p_i, q_i^+\right), \left(\eta_i, \mathcal{P}_i^+\right), \left(\mathcal{P}_i, \eta_i^+\right), \right.$

$\big(\eta_{lm}, \mathcal{P}^+_{lm}\big)$, $\big(\mathcal{P}_{lm}, \eta^+_{lm}\big)$, $(\eta, \mathcal{P}^+)$, $(\mathcal{P}, \eta^+)\big\}$, and a set of zero-mode pairs, $\big\{(q_0, p_0)$, $(\eta_0, \mathcal{P}_0)$, $\big(\eta^i_{\mathcal{H}}, \mathcal{P}^i_{\mathcal{H}}\big)\big\}$. Following [34], they obey the nonvanishing (anti)commutation relations

$$\{\eta, \mathcal{P}^+\} = \{\mathcal{P}, \eta^+\} = \{\eta_i, \mathcal{P}^+_i\} = \{\mathcal{P}_i, \eta^+_i\} = 1\,, \quad \{\eta_{lm}, \mathcal{P}^+_{lm}\} = \{\mathcal{P}_{lm}, \eta^+_{lm}\} = 1\,, \tag{4.5}$$
$$[q_i, p^+_i] = [p_i, q^+_i] = 1\,, \qquad [q_0, p_0] = \{\eta_0, \mathcal{P}_0\} = \{\eta^i_{\mathcal{H}}, \mathcal{P}^i_{\mathcal{H}}\} = \imath\,;$$

they also possess the standard ghost number distribution, $gh(\mathcal{C}^i) = -gh(\mathcal{P}_i) = 1$, providing the property $gh(\tilde{Q}') = 1$, and have the Hermitian conjugation properties of zero-mode pairs,[5]

$$\big(q_0, \eta_0, \eta^i_{\mathcal{H}}, p_0, \mathcal{P}_0, \mathcal{P}^i_{\mathcal{H}}\big)^+ = \big(q_0, \eta_0, \eta^i_{\mathcal{H}}, p_0, -\mathcal{P}_0, -\mathcal{P}^i_{\mathcal{H}}\big)\,. \tag{4.6}$$

The property of the BRST operator to be Hermitian is defined by the rule

$$Q'^+ K = K Q'\,, \tag{4.7}$$

and is calculated with respect to the scalar product $\langle\ |\ \rangle$ in $\mathcal{H}_{\rm tot}$ with the measure d^dx, which, in its turn, is constructed as the direct product of the scalar products in $\mathcal{H}, \mathcal{H}'$ and $\mathcal{H}_{gh}$. The operator K in (4.7) is the tensor product of the operator K' in $\mathcal{H}'$ and the unit operators in $\mathcal{H}$, $\mathcal{H}_{gh}$

$$K = \hat{1} \otimes K' \otimes \hat{1}_{gh}\,. \tag{4.8}$$

Thus, we have constructed a Hermitian BRST operator for the entire superalgebra of O_I. In the next section, this operator will be used to construct a Lagrangian action for fermionic HS fields of spin (s_1, s_2) in a flat space.

5. Construction of Lagrangian actions

The construction of Lagrangians for fermionic higher-spin fields in a d-dimensional Minkowski space can be developed by partially following the algorithm of [44], which is a particular case of our construction, corresponding to $n_2 = 0$. As a first step, we extract the dependence of the BRST operator Q' (4.4) on the ghosts $\eta^i_{\mathcal{H}}, \mathcal{P}^i_{\mathcal{H}}$, so as to obtain the BRST operator Q only for the system of converted first-class constraints $\{O_I\} \setminus \{\mathcal{H}^i\}$:

$$Q' = Q + \eta^i_{\mathcal{H}}(\sigma^i + h^i) + \mathcal{A}^i \mathcal{P}^i_{\mathcal{H}}\,, \tag{5.1}$$

where

$$\mathcal{A}^i = \imath\Big\{\eta^+_{ii}\eta_{ii} - 2q_i q^+_i + \eta\eta^+(\delta^{i1} - \delta^{i2}) + \frac{1}{4}\eta^+_{12}\eta_{12}(\delta^{i1} + \delta^{i2})\Big\}. \tag{5.2}$$

[5] In terms of the redefinition $\big(p_i, \mathcal{P}_0, \mathcal{P}^i_{\mathcal{H}}\big) \mapsto \imath\,\big(p_i, \mathcal{P}_0, \mathcal{P}^i_{\mathcal{H}}\big)$, the BRST operator (4.4) and relations (4.5) are written in the notation of [44, 45].

$$
\begin{aligned}
Q = {} & q_0 T_0 + q_i^+ T^i + T_i^+ q^i + \eta_0 L_0 + \eta_i^+ L^i + L_i^+ \eta^i + \eta_{lm}^+ L^{lm} + L_{lm}^+ \eta^{lm} + \eta^+ T + T^+ \eta \\
& + i(\eta_i^+ q_i - \eta_i q_i^+) p_0 + \eta_{ii} q_i^+ p_i^+ + \eta_{ii}^+ q_i p_i - i(q_0^2 - \eta_i^+ \eta_i)\mathcal{P}_0 + (\eta_{ii}^+ \eta_i - 2 q_0 q_i^+)\mathcal{P}_i \\
& + (\eta_i^+ \eta_{ii} - 2 q_0 q_i)\mathcal{P}_i^+ - 2 q_i^{+2} \mathcal{P}_{ii} - 2 q_i^2 \mathcal{P}_{ii}^+ \\
& - 2\left[2 q_1 q_2 - \eta^+ \eta_{22} - \eta \eta_{11}\right] \mathcal{P}_{12}^+ + \eta^+ \eta_{12} \mathcal{P}_{11}^+ + \eta \eta_{12} \mathcal{P}_{22}^+ \\
& - 2\left[2 q_1^+ q_2^+ + \eta^+ \eta_{11}^+ + \eta \eta_{22}^+\right] \mathcal{P}_{12} - \eta \eta_{12}^+ \mathcal{P}_{11} - \eta^+ \eta_{12}^+ \mathcal{P}_{22} \\
& + \left[\frac{1}{2}\eta_{12}^+ \eta_{11} + \frac{1}{2}\eta_{22}^+ \eta_{12} - 2 q_1 q_2^+\right] \mathcal{P} + \left[\frac{1}{2}\eta_{11}^+ \eta_{12} + \frac{1}{2}\eta_{12}^+ \eta_{22} - 2 q_2 q_1^+\right] \mathcal{P}^+ \\
& + \left[\frac{1}{2} q_1 \eta_{12}^+ - \eta^+ q_1^+\right] p_2 + \left[\frac{1}{2} q_1^+ \eta_{12} - \eta q_1\right] p_2^+ + \left[\frac{1}{2} q_2 \eta_{12}^+ - \eta q_2^+\right] p_1 \\
& + \left[\frac{1}{2} q_2^+ \eta_{12} - \eta^+ q_2\right] p_1^+ + \left[\frac{1}{2}\eta_{12}^+ \eta_2 - \eta \eta_2^+\right] \mathcal{P}_1 + \left[\frac{1}{2}\eta_2^+ \eta_{12} + \eta^+ \eta_2\right] \mathcal{P}_1^+ \\
& + \left[\frac{1}{2}\eta_{12}^+ \eta_1 - \eta^+ \eta_1^+\right] \mathcal{P}_2 + \left[\frac{1}{2}\eta_1^+ \eta_{12} - \eta_1 \eta\right] \mathcal{P}_2^+ \,, \qquad (5.3)
\end{aligned}
$$

The generalized spin operator $\sigma^i = (\sigma^1, \sigma^2)$, extended by the ghost Wick-pair variables, has the form

$$
\begin{aligned}
\sigma^i = {} & \mathcal{H}^i - h^i + q_i p_i^+ + q_i^+ p_i - \eta_i \mathcal{P}_i^+ + \eta_i^+ \mathcal{P}_i - 2\eta_{ii} \mathcal{P}_{ii}^+ + 2\eta_{ii}^+ \mathcal{P}_{ii} \\
& (\delta^{i1} + \delta^{i2})[\eta_{12}^+ \mathcal{P}_{12} - \eta_{12} \mathcal{P}_{12}^+] + (\delta^{i2} - \delta^{i1})[\eta^+ \mathcal{P} - \eta \mathcal{P}^+] \,. \qquad (5.4)
\end{aligned}
$$

Second, we choose a representation of the Hilbert space permitting us to find the BRST cohomology spaces for the first-class constraint system,

$$
\left(p_0, q_i, p_i, \mathcal{P}_0, \mathcal{P}_{\mathcal{H}}^i, \eta_i, \mathcal{P}_i, \eta_{lm}, \mathcal{P}_{lm}, \eta, \mathcal{P}\right) |0\rangle = 0, \qquad (5.5)
$$

and to extract from $\mathcal{H}_{\rm tot}$ the Hilbert subspace that does not depend on the $\eta_{\mathcal{H}}^i$ operators (since $\mathcal{H}^i$ are not first-class constraints as the other O_I),

$$
\begin{aligned}
|\chi\rangle = {} & \sum_{k_r} (q_0)^{k_1} (q_i^+)^{k_{2i}} (p_i^+)^{k_{3i}} (\eta_0)^{k_4} (f_i^+)^{k_{5i}} (\eta_i^+)^{k_{6i}} (\mathcal{P}_i^+)^{k_{7i}} (\eta_{lm}^+)^{k_{lm}^8} (\mathcal{P}_{lm}^+)^{k_{lm}^9} (\eta^+)^{k_{10}} (\mathcal{P}^+)^{k_{11}} \times \\
& \times (b_{no}^+)^{k_{no}^{12}} (b^+)^{k_{13}} a_1^{+\mu_1} \ldots a_1^{+\mu_{k_{14}}} a_2^{+\nu_1} \ldots a_2^{+\nu_{k_{15}}} \chi_{(\mu)_{k_{14}},(\nu)_{k_{15}}}^{k_1 k_{2i} \ldots k_{13}}(x) |0\rangle. \qquad (5.6)
\end{aligned}
$$

The sum in (5.6) is taken over k_1, k_{2i}, k_{3i}, k_{no}^{12}, k_{13}, k_{14}, k_{15}, running from 0 to infinity, for $i, l, m, n, o = 1, 2$, $l \leq m, n \leq o$, and over the other indices, running from 0 to 1. Next, we derive from the equations determining the physical vector, $Q'|\chi\rangle = 0$, and from the reducible gauge transformations, $\delta|\chi\rangle = Q'|\Lambda\rangle$, $\delta|\Lambda\rangle = Q'|\Lambda^{(1)}\rangle$, …, $\delta|\Lambda^{(s-1)}\rangle = Q'|\Lambda^{(s)}\rangle$, a sequence of relations:

$$
\begin{aligned}
& Q|\chi\rangle = 0, && (\sigma^i + h^i)|\chi\rangle = 0, && (\varepsilon, gh)\,(|\chi\rangle) = (1, 0), && (5.7) \\
& \delta|\chi\rangle = Q|\Lambda\rangle, && (\sigma^i + h^i)|\Lambda\rangle = 0, && (\varepsilon, gh)\,(|\Lambda\rangle) = (0, -1), && (5.8) \\
& \delta|\Lambda\rangle = Q|\Lambda^{(1)}\rangle, && (\sigma^i + h^i)|\Lambda^{(1)}\rangle = 0, && (\varepsilon, gh)\,(|\Lambda^{(1)}\rangle) = (1, -2), && (5.9) \\
& \delta|\Lambda^{(s-1)}\rangle = Q|\Lambda^{(s)}\rangle, && (\sigma^i + h^i)|\Lambda^{(s)}\rangle = 0, && (\varepsilon, gh)\,(|\Lambda^{(s)}\rangle) = (s, -s-1). && (5.10)
\end{aligned}
$$

The middle set of equations in (5.7)–(5.10) determines the possible values of the parameters h^i and the eigenvectors of the operators σ^i. Solving these equations, we obtain a set of eigenvectors, $|\chi\rangle_{(n_1,n_2)}$, $|\Lambda\rangle_{(n_1,n_2)}$, …, $|\Lambda^{(s)}\rangle_{(n_1,n_2)}$, $n_1 \geq n_2 \geq 0$, and a set of eigenvalues,

$$-h^i = n^i + \frac{d-4}{2} - \delta^{2i}2\,, \quad i = 1,2\,, \quad n_1 \in \mathbb{Z}, n_2 \in \mathbb{N}_0\,, \tag{5.11}$$

with (n_1, n_2) related to spin, $\mathbf{s} = (n_1, n_2) + (1/2, 1/2)$. The values of n_1, n_2 are related to the spin components s_1, s_2 of the field, because the proper vector $|\chi\rangle_{(n_1,n_2)}$ corresponding to (h_1, h_2) has the leading term $\chi^{0\cdots0}_{(\mu)_{n_1},(\nu)_{n_2}}(x)$, independent of the auxiliary and ghost operators, which corresponds to the field $\Phi_{(\mu)_{n_1},(\nu)_{n_2}}(x)$ with the initial value of spin $\mathbf{s} = (s_1, s_2)$ in the decomposition (5.6),

$$\begin{aligned}|\chi\rangle_{(n_1,n_2)} = \Big[&a_1^{+\mu_1}\ldots a_1^{+\mu_{n_1}} a_2^{+\nu_1}\ldots a_2^{+\nu_{n_2}}\chi^{0\cdots0}_{(\mu)_{n_1},(\nu)_{n_2}}(x) \\ &+b^+a^{+\mu_1}\ldots a^{+\mu_{n_1+1}}a_2^{+\nu_1}\ldots a_2^{+\nu_{n_2-1}}\chi^{0\cdots01}_{(\mu)_{n_1+1},(\nu)_{n_2-1}}(x) \\ &+b^+_{12}a^{+\mu_1}\ldots a^{+\mu_{n_1-1}}a_2^{+\nu_1}\ldots a_2^{+\nu_{n_2-1}}\chi^{0\cdots0100}_{(\mu)_{n_1-1},(\nu)_{n_2-1}}(x)+\ldots\Big]|0\rangle,\end{aligned} \tag{5.12}$$

where the values of (n_1, n_2) can be composed of the set of coefficients $\{k_r\}\setminus\{k_1, k_4\}$ in (5.6) by the formulae

$$\begin{aligned}n_i = {}& k_{2i} + k_{3i} + k_{5i} + k_{6i} + k_{7i} + 2k^8_{ii} + k^8_{12} + 2k^9_{ii} + k^9_{12} \\ & 2k^{12}_{ii} + k^{12}_{12} + (-1)^i(k_{10} + k_{11} + k_{13}) + k_{14}\delta_{i1} + k_{15}\delta_{i2}\,.\end{aligned} \tag{5.13}$$

Therefore, relations (5.7)–(5.10) guarantee both the extraction of vectors with the required value of spin and the nilpotency of Q in the corresponding Hilbert subspace. If one fixes the value of spin, then the parameters h^i are also fixed by (5.11). Having fixed the value of h^i, we should substitute it into each of the expressions (5.7)–(5.10).

Third, we should extract the zero-mode ghosts from the operator Q as follows:

$$Q = q_0\tilde{T}_0 + \eta_0 L_0 + \imath(\eta_i^+ q_i - \eta_i q_i^+)p_0 - \imath(q_0^2 - \eta_i^+\eta_i)\mathcal{P}_0 + \Delta Q, \tag{5.14}$$

where

$$\tilde{T}_0 = T_0 - 2q_i^+\mathcal{P}_i - 2q_i\mathcal{P}_i^+\,, \tag{5.15}$$

$$\begin{aligned}\Delta Q = {}& q_i^+T^i + T_i^+q^i + \eta_i^+L^i + L_i^+\eta^i + \eta^+_{lm}L^{lm} + L^+_{lm}\eta^{lm} + \eta^+T + T^+\eta \\ & +\eta_{ii}q_i^+p_i^+ + \eta^+_{ii}q_ip_i + \eta^+_{ii}\eta_i\mathcal{P}_i + \eta_i^+\eta_{ii}\mathcal{P}_i^+ - 2q_i^{+2}\mathcal{P}_{ii} - 2q_i^2\mathcal{P}^+_{ii} \\ & -2\left[2q_1q_2 - \eta^+\eta_{22} - \eta\eta_{11}\right]\mathcal{P}^+_{12} + \eta^+\eta_{12}\mathcal{P}^+_{11} + \eta\eta_{12}\mathcal{P}^+_{22} \\ & -2\left[2q_1^+q_2^+ + \eta^+\eta^+_{11} + \eta\eta^+_{22}\right]\mathcal{P}_{12} - \eta\eta^+_{12}\mathcal{P}_{11} - \eta^+\eta^+_{12}\mathcal{P}_{22} \\ & +\left[\frac{1}{2}\eta^+_{12}\eta_{11} + \frac{1}{2}\eta^+_{22}\eta_{12} - 2q_1q_2^+\right]\mathcal{P} + \left[\frac{1}{2}\eta^+_{11}\eta_{12} + \frac{1}{2}\eta^+_{12}\eta_{22} - 2q_2q_1^+\right]\mathcal{P}^+ \\ & +\left[\frac{1}{2}q_1\eta^+_{12} - \eta^+q_1^+\right]p_2 + \left[\frac{1}{2}q_1^+\eta_{12} - \eta q_1\right]p_2^+ + \left[\frac{1}{2}q_2\eta^+_{12} - \eta q_2^+\right]p_1 \\ & +\left[\frac{1}{2}q_2^+\eta_{12} - \eta^+q_2\right]p_1^+ + \left[\frac{1}{2}\eta^+_{12}\eta_2 - \eta\eta_2^+\right]\mathcal{P}_1 + \left[\frac{1}{2}\eta_2^+\eta_{12} + \eta^+\eta_2\right]\mathcal{P}_1^+ \\ & +\left[\frac{1}{2}\eta^+_{12}\eta_1 - \eta^+\eta_1^+\right]\mathcal{P}_2 + \left[\frac{1}{2}\eta_1^+\eta_{12} - \eta_1\eta\right]\mathcal{P}_2^+\,.\end{aligned} \tag{5.16}$$

Here, $\tilde{T}_0$, ΔQ are independent of q_0, p_0, η_0, $\mathcal{P}_0$. We also expand the state vector and gauge parameters in powers of the zero-mode ghosts:

$$|\chi\rangle = \sum_{k=0}^{\infty} q_0^k(|\chi_0^k\rangle + \eta_0|\chi_1^k\rangle), \qquad gh(|\chi_m^k\rangle) = -(m+k), \tag{5.17}$$

$$|\Lambda^{(s)}\rangle = \sum_{k=0}^{\infty} q_0^k(|\Lambda^{(s)k}_0\rangle + \eta_0|\Lambda^{(s)k}_1\rangle), \qquad gh(|\Lambda^{(s)k}_m\rangle) = -(s+k+m+1). \tag{5.18}$$

Following the procedure described in [24, 44], we get rid of all the fields except two, $|\chi_0^0\rangle$, $|\chi_0^1\rangle$.

Namely, after the extraction of zero-mode ghosts from the BRST operator Q (5.14), as well as from the state vector and the gauge parameter (5.17), (5.18), the gauge transformation for the fields $|\chi_0^k\rangle$, $k \geq 2$ has the form

$$\delta|\chi_0^k\rangle = \Delta Q|\Lambda_0^k\rangle + \eta_i\eta_i^+|\Lambda_1^k\rangle + (k+1)(q_i\eta_i^+ - \eta_i q_i^+)|\Lambda_0^{k+1}\rangle + \tilde{T}_0|\Lambda_0^{k-1}\rangle + |\Lambda_1^{k-2}\rangle\,, \tag{5.19}$$

implying, by induction, that we can make all the fields $|\chi_0^k\rangle$, $k \geq 2$ equal to zero by using the gauge parameters $|\Lambda_1^k\rangle$. Then, considering the equations of motion for the powers $q_0^k, k \geq 3$ and taking into account that $|\chi_0^k\rangle = 0$, $k \geq 2$, we can see that these equations contain the subsystem

$$|\chi_1^{k-2}\rangle = \eta_i\eta_i^+|\chi_1^k\rangle\,, \qquad k \geq 3\,, \tag{5.20}$$

which permits us to find, by induction, that all the fields $|\chi_1^k\rangle$, $k \geq 1$ are equal to zero. Finally, we examine the equations of motion for the power q_0^2:

$$|\chi_1^0\rangle = -\tilde{T}_0|\chi_0^1\rangle\,, \tag{5.21}$$

in order to express the vector $|\chi_1^0\rangle$ in terms of $|\chi_0^1\rangle$. Thus, as in the totally symmetric case, there remain only two independent fields: $|\chi_0^0\rangle$, $|\chi_0^1\rangle$. The first equation in (5.7), (5.14), the decomposition (5.17), and the above analysis then imply that the independent equations of motion for these vectors have the form

$$\Delta Q|\chi_0^0\rangle + \frac{1}{2}\{\tilde{T}_0, \eta_i^+\eta_i\}|\chi_0^1\rangle = 0, \tag{5.22}$$

$$\tilde{T}_0|\chi_0^0\rangle + \Delta Q|\chi_0^1\rangle = 0\,, \tag{5.23}$$

where $\{F, G\} = FG + GF$ for any quantities F, G.

Then, due to the fact that the operators Q, $\tilde{T}_0$, $\eta_i^+\eta_i$ commute with σ^i, we obtain from (5.22), (5.23) the equations of motion for the fields with a fixed value of spin:

$$\Delta Q|\chi_0^0\rangle_{(n_1,n_2)} + \frac{1}{2}\{\tilde{T}_0, \eta_i^+\eta_i\}|\chi_0^1\rangle_{(n_1,n_2)} = 0, \tag{5.24}$$

$$\tilde{T}_0|\chi_0^0\rangle_{(n_1,n_2)} + \Delta Q|\chi_0^1\rangle_{(n_1,n_2)} = 0. \tag{5.25}$$

where the fields $|\chi_0^k\rangle_{(n_1,n_2)}$, $k = 0, 1$ are assumed to obey the relations

$$\sigma^i|\chi_0^k\rangle_{(n_1,n_2)} = \big(n_i + (d-4)/2 - 2\delta_{i2}\big)|\chi_0^k\rangle_{(n_1,n_2)}\,, \qquad k = 0, 1. \tag{5.26}$$

The field equations (5.24), (5.25) are Lagrangian ones and can be deduced, in view of the invertibility of the operator K, from the following Lagrangian action:[6]

$$\mathcal{S}_{(n_1,n_2)} = {}_{(n_1,n_2)}\langle\tilde{\chi}^0_0|K_{(n_1,n_2)}\tilde{T}_0|\chi^0_0\rangle_{(n_1,n_2)} + \frac{1}{2}\,{}_{(n_1,n_2)}\langle\tilde{\chi}^1_0|K_{(n_1,n_2)}\{\tilde{T}_0,\eta^+_i\eta_i\}|\chi^1_0\rangle_{(n_1,n_2)}\,,$$
$$+{}_{(n_1,n_2)}\langle\tilde{\chi}^0_0|K_{(n_1,n_2)}\Delta Q|\chi^1_0\rangle_{(n_1,n_2)} + {}_{(n_1,n_2)}\langle\tilde{\chi}^1_0|K_{(n_1,n_2)}\Delta Q|\chi^0_0\rangle_{(n_1,n_2)}\,, \quad (5.27)$$

where the standard scalar product for the creation and annihilation operators is assumed, and $K_{(n_1,n_2)}$ is the operator K (4.8) with the following substitution: $h^i \to -\big(n_i + (d-4)/2 - 2\delta_{i2}\big)$.

The equations of motion (5.24), (5.25) and the action (5.27) are invariant with respect to the gauge transformations

$$\delta|\chi^0_0\rangle_{(n_1,n_2)} = \Delta Q|\Lambda^0_0\rangle_{(n_1,n_2)} + \frac{1}{2}\{\tilde{T}_0,\eta^+_i\eta_i\}|\Lambda^1_0\rangle_{(n_1,n_2)}\,, \quad (5.28)$$

$$\delta|\chi^1_0\rangle_{(n_1,n_2)} = \tilde{T}_0|\Lambda^0_0\rangle_{(n_1,n_2)} + \Delta Q|\Lambda^1_0\rangle_{(n_1,n_2)}\,, \quad (5.29)$$

which are reducible, with the gauge parameters $|\Lambda^{(s)j}{}_0\rangle_{(n_1,n_2)}$, $j = 0, 1$ subject to the same conditions as those for $|\chi^j_0\rangle_{(n_1,n_2)}$ in (5.26),

$$\delta|\Lambda^{(s)0}{}_0\rangle_{(n_1,n_2)} = \Delta Q|\Lambda^{(s+1)0}{}_0\rangle_{(n_1,n_2)} + \frac{1}{2}\{\tilde{T}_0,\eta^+_i\eta_i\}|\Lambda^{(s+1)1}{}_0\rangle_{(n_1,n_2)}, \quad |\Lambda^{(0)0}{}_0\rangle = |\Lambda^0_0\rangle\,, \quad (5.30)$$

$$\delta|\Lambda^{(s)1}{}_0\rangle_{(n_1,n_2)} = \tilde{T}_0|\Lambda^{(s+1)0}{}_0\rangle_{(n_1,n_2)} + \Delta Q|\Lambda^{(s+1)1}{}_0\rangle_{(n_1,n_2)}, \quad |\Lambda^{(0)1}{}_0\rangle = |\Lambda^1_0\rangle\,, \quad (5.31)$$

and with a finite number of reducibility stages[7] at $s_{\max} = n_1 + n_2$ for spin $\mathbf{s} = (n_1 + 1/2, n_2 + 1/2)$.

In addition to restrictions (5.13), the set of coefficients $\{k_r\}\setminus\{k_1, k_4\}$ in (5.6) for fixed values of n_i, satisfies the following equations for $|\chi^j_0\rangle_{(n_1,n_2)}$ $|\Lambda^{(s)j_0}\rangle_{(n_1,n_2)}$, respectively:

$$|\chi^j_0\rangle_{(n_1,n_2)} : \sum_i (k_{2i} - k_{3i} + k_{6i} - k_{7i}) + \sum_{l\le m}\big(k^8_{lm} - k^9_{lm}\big) + k_{10} - k_{11} = -j, \quad (5.32)$$

$$|\Lambda^{(s)j}_0\rangle_{(n_1,n_2)} : \sum_i (k_{2i} - k_{3i} + k_{6i} - k_{7i}) + \sum_{l\le m}\big(k^8_{lm} - k^9_{lm}\big) + k_{10} - k_{11} = -(s+j+1), \quad (5.33)$$

due to the ghost number distribution (5.17), (5.18).

Thus, we have constructed, by using the BRST procedure, a gauge-invariant Lagrangian description of fermionic fields with a mixed symmetry of any fixed spin $\mathbf{s}$.

A Lagrangian description of a theory with a half-integer mixed-symmetry HS field of mass m in a d-dimensional Minkowski space is deduced by dimensional reduction of a massless HS field theory of the same type in a $(d+1)$-dimensional flat space $\mathbb{R}^{1,d}$. To this end, we have to make a projection $\mathbb{R}^{1,d} \to \mathbb{R}^{1,d-1}$ with a simple decomposition,

$$\partial^M = (\partial^\mu, \imath m)\,, \qquad a^M_i = (a^\mu_i, b_i)\,, \qquad a^{M+}_i = (a^{\mu+}_i, b^+_i)\,, \quad \tilde{\gamma}^M = (\tilde{\gamma}^\mu, \tilde{\gamma})\,, \quad (5.34)$$

$$M = 0, 1, \ldots, d\,, \qquad \mu = 0, 1, \ldots, d-1\,, \quad \eta^{MN} = diag(1, -1, \ldots, -1, -1)\,, \quad (5.35)$$

[6] As usual, the action is defined up to an overall factor.

[7] In the case of a spin-tensor field $\Phi_{(\mu)_{n_1},(\nu)_{n_2},\ldots,(\rho)_{n_k}}(x)$ with the Young tableaux having k rows, one can show that the stage of reducibility for the corresponding Lagrangian formulation must be equal to $s_{\max} = \sum_{l=1}^k n_l + k(k-1)/2 - 1$, so that for a totally symmetric field $\Phi_{(\mu)_{n_1}}(x)$ one has $s_{\max} = n_1 - 1$, in accordance with [44].

whereas the quantities $\tilde{\gamma}T_0\tilde{\gamma}$, identical with T_0 for massless HS fields, transform, for massive fields, as $\tilde{\gamma}T_0\tilde{\gamma} = T_0^*$, $T_0^* = -\imath\tilde{\gamma}^\mu\partial_\mu - m$. Thus, for the coefficient functions entering with $\tilde{\gamma}$-matrix into the decomposition of any vector composed of $|\chi_0^j\rangle_{(n_1,n_2)}$, $|\Lambda_0^{(s)j}\rangle_{(n_1,n_2)}$, the homogeneous part of a vector $|\Xi\rangle_{(n_1,n_2)}$ with respect to $\tilde{\gamma}$-matrix is subject to

$$\imath\gamma^M\partial_M|\Xi\rangle = \Big(\imath\gamma^\mu\partial_\mu - (-1)^{deg_{\tilde{\gamma}}|\Xi\rangle}m\Big)|\Xi\rangle, \quad deg_{\tilde{\gamma}}|\Xi\rangle = 0,1\,, \quad |\Xi\rangle \in \{|\chi_0^j\rangle, |\Lambda_0^{(s)j}\rangle\}\,. \quad (5.36)$$

Relations (5.34), (5.35) indicate the presence of four additional second-class constraints, l_i, l_i^+, with corresponding oscillator operators b_i, b_i^+, $[b_i, b_j^+] = \delta_{ij}$, in comparison with the massless case.[8]

A simultaneous construction of Lagrangian actions describing the propagation of all massless (massive) fermionic fields with two rows of the Young tableaux in Minkowski space is analogous to the case of totally symmetric spin-tensors in flat space [44], and we only note that a necessary condition for solving this problem is to replace in Q', Q, K the parameters $-h^i$ by the operators σ^i in an appropriate way and discard condition (5.26) for the fields and gauge parameters. Among other things, this completes the conversion procedure for the initial constraint system $\{o_I\} \setminus \{g_0^i\}$ and the construction of a nilpotent BRST operator in the entire Hilbert space for the set of converted constraints $\{O_I\} \setminus \{\mathcal{H}^i\}$.

In the next section, we shall sketch a proof of the fact that the action, in fact, reproduces the correct equations of motion (2.2)–(2.5).

6. Reduction to the initial irreducible relations

Let us briefly show the fact that it is only the solutions of the equations of motion (2.2)–(2.5) that determine the space of BRST cohomologies of the operator Q (5.3) with a vanishing ghost number in the Fock space $\mathcal{H}$ for the basic fermionic field with spin $\mathbf{s} = (n_1+1/2, n_2+1/2)$. To solve this problem, we can follow two ways: the first one is realized, for instance, in [26] for massless totally-symmetric bosonic fields in a flat space-time, and the second one, for totally-symmetric fermionic fields [44, 45]. We will use the technics of [44, 45], taking into account the fact that the spectrum of component fields for an arbitrary vector $|\chi\rangle_{(n_1,n_2)}$ in (5.6) for $k_1 = k_4 = 0$ is essentially larger than the spectrum for a totally-symmetric fermionic vector $|\chi\rangle_{(n_1+n_2,0)}$, for whose description one should not use the operators q_2^+, p_2^+, f_2^+, η_2^+, $\mathcal{P}_2^+$, η_{12}^+, $\mathcal{P}_{12}^+$, η_{22}^+, $\mathcal{P}_{22}^+$, η^+, $\mathcal{P}^+$, b^+, b_{12}^+, b_{22}^+, $a_2^{+\nu}$ and the corresponding conjugations.[9] As a consequence, the character of proof is more involved even in comparison with the case of the AdS space [45].

In the standard manner, the proof consists of two steps. First, in order to simplify the spectrum of the gauge parameters $|\Lambda^{(s)j}_0\rangle$ and the fields $|\chi_0^j\rangle$, $j = 0, 1$, we apply to them a

[8] On a basis of the above arguments, one can state that the procedure of dimensional reduction given by relations (5.34), (5.35) can be applied to massless mixed-symmetry bosonic fields [9] in order to obtain a Lagrangian description of massive mixed-symmetry bosonic fields, whereas for fermionic HS fields it is necessary to add eqs. (5.36), Cf. [44].

[9] The total number of independent "creation" operators which are necessary to compose the vector $|\chi\rangle_{(n_1,n_2)}$ is more than twice as large as the number required for $|\chi\rangle_{(n_1+n_2,0)}$: $(2d+22)/(d+8)$.

gauge-fixing based on the structure of gauge transformations (5.28)–(5.31) and extract the physical field $|\Phi\rangle_{(n_1,n_2)}$ alone, by using (2.7) for $s = (n_1 + 1/2, n_2 + 1/2)$:

$$|\chi^0_0\rangle_{(n_1,n_2)} = |\Phi\rangle_{(n_1,n_2)} + |\Phi_A\rangle_{(n_1,n_2)}, \qquad |\Phi_A\rangle_{(n_1,n_2)}\Big|_{\mathcal{C}=\mathcal{P}=b^+_{ij}=b^+=f^+_i=0} = 0. \tag{6.1}$$

Second, we use a part of the Lagrangian equations of motion (5.24), (5.25) in order to select from them only the equations of motion for $|\Phi\rangle_{(n_1,n_2)}$, and to remove all of the remaining auxiliary fields of lower spins.

Let us now describe the basic sequence of gauge-fixing. Our strategy consists in a successive elimination of the terms with $\mathcal{P}^+_{11}$ from the fields $|\chi^j_0\rangle$ and gauge parameters $|\Lambda^{(s)\,j}_0\rangle$, starting from the top of the tower of gauge transformations (5.28)–(5.31). For this purpose, it should be noted that we have a reducible gauge theory of $(n_1 + n_2)$-th stage of reducibility. Because of the restrictions for the spin (5.13) and ghost number (5.32), (5.33), the independent parameters of the lowest stage have the form

$$|\Lambda^{(n_1+n_2)0}_0\rangle_{(n_1,n_2)} \;=\; (p^+_1)^{n_1}\,\big|A(p^+_i,\mathcal{P}^+_i,\mathcal{P}^+)\rangle_{(0,n_2)}\;, \quad |\Lambda^{(n_1+n_2)1}_0\rangle_{(n_1,n_2)} \;\equiv\; 0\,, \tag{6.2}$$

where the vector $\big|A(p^+_i,\mathcal{P}^+_i,\mathcal{P}^+)\rangle_{(l,n_2-m)} \equiv \big|A\rangle_{(l,n_2-m)}$, $l < n_1$, $m < n_2$, has the structure

$$\begin{aligned}\big|A\rangle_{(l,n_2-m)} \;=\; \mathcal{P}^+\Bigg\{ &\sum_{k=m+1}^{n_2} (p^+_1)^k (p^+_2)^{n_2-k} |\omega^{1l}_k\rangle_{(-k+l+1,k-1-m)} \\ &+\mathcal{P}^+_1 \sum_{k=m}^{n_2-1} (p^+_1)^k (p^+_2)^{n_2-k-1} |\omega^{2l}_k\rangle_{(-k+l,k-m)} \\ &+\mathcal{P}^+_2 \sum_{k=m+1}^{n_2-1} (p^+_1)^k (p^+_2)^{n_2-k-1} |\omega^{3l}_k\rangle_{(-k+l+1,k-1-m)} \\ &+\mathcal{P}^+_1\mathcal{P}^+_2 \sum_{k=m}^{n_2-2} (p^+_1)^k (p^+_2)^{n_2-k-2} |\omega^{4l}_k\rangle_{(-k+l,k-m)} \Bigg\}.\end{aligned} \tag{6.3}$$

It can be verified directly that one can eliminate the dependence on the ghost $\mathcal{P}^+_{11}$ from the gauge function $|\Lambda^{(n_1+n_2-1)0}_0\rangle$ of (n_1+n_2-1)-th stage of reducibility, whereas the vector $|\Lambda^{(n_1+n_2-1)1}_0\rangle$ has the same structure as $|\Lambda^{(n_1+n_2)0}_0\rangle$ in (6.2). Indeed, for $|\Lambda^{(n_1+n_2-1)0}_0\rangle$ we have the following expansion in the powers of $\mathcal{P}^+_{11}$, $\mathcal{P}^+_{12}$, $\mathcal{P}^+_{22}$:

$$\begin{aligned}|\Lambda^{(n_1+n_2-1)0}_0\rangle_{(n_1,n_2)} \;=\; &|\Lambda^{(n_1+n_2-1)0}_{00}\rangle_{(n_1,n_2)} + \mathcal{P}^+_{11}(p^+_1)^{n_1-2}|\tilde{A}\rangle_{(0,n_2)}\,, \\ &+\mathcal{P}^+_{12}(p^+_1)^{n_1-1}|\tilde{A}\rangle_{(0,n_2-1)} + \mathcal{P}^+_{22}(p^+_1)^{n_1}|\tilde{A}\rangle_{(0,n_2-2)}\,,\end{aligned} \tag{6.4}$$

with $|\tilde{A}\rangle_{(0,n_2-k)}$ defined according to (6.4), so that the gauge transformation (5.30) at $\mathcal{P}^+_{11}$ implies

$$\delta|\tilde{A}\rangle_{(0,n_2)} = -2q^2_1(p^+_1)^2|A\rangle_{(0,n_2)}\,. \tag{6.5}$$

After the vector $|\tilde{A}\rangle_{(0,n_2)}$ has been removed, the theory is transformed to a theory of $(n_1 + n_2 - 1)$-th stage of reducibility. Then, it is possible to verify that one can remove the dependence of $|\Lambda^{(n_1+n_2-2)\,j}_0\rangle$ on $\mathcal{P}^+_{11}$ with the help of the remaining gauge parameters $|\Lambda^{(n_1+n_2-1)\,j}_0\rangle$, which do not depend on $\mathcal{P}^+_{11}$.

It then becomes possible to prove by induction that after removing the dependence on $\mathcal{P}^+_{11}$ from the gauge parameters up to the $(s+1)$-th stage, $|\Lambda^{(l)k}_0\rangle$, $k=0,1$, $l \geq s+1$ (i.e., we have $\eta_{11}|\Lambda^{(l)k}_0\rangle = 0$), and applying the restricted vector $|\Lambda^{(s+1)k}_0\rangle$, one can eliminate the dependence on $\mathcal{P}^+_{11}$ from the gauge functions $|\Lambda^{(s)k}_0\rangle$. To this end, we introduce the following notation for the gauge parameters related to their expansion in the ghosts $\mathcal{P}^+_{ij}$:

$$|\Lambda^{(l)k}_0\rangle = |\Lambda^{(l)k}_{00}\rangle + \sum_{i\leq j}\mathcal{P}^+_{ij}|\Lambda^{(l)k}_{0ij}\rangle + \mathcal{P}^+_{11}\Big(\mathcal{P}^+_{12}|\Lambda^{(l)k}_{01}\rangle + \mathcal{P}^+_{22}|\Lambda^{(l)k}_{02}\rangle + \mathcal{P}^+_{12}\mathcal{P}^+_{22}|\Lambda^{(l)k}_{03}\rangle\Big) + \mathcal{P}^+_{12}\mathcal{P}^+_{22}|\Lambda^{(l)k}_{04}\rangle\,. \tag{6.6}$$

Here and elsewhere, we omit the vector subscripts associated with the eigenvalues of the operators σ^i (5.26). From (5.30), (5.31), we obtain the gauge transformations for $|\Lambda^{(s)k}_{011}\rangle$, $|\Lambda^{(s)k}_{0p}\rangle$, $p=1,2,3$, being the coefficients at $\mathcal{P}^+_{11}$, namely,

$$\delta|\Lambda^{(s)k}_{011}\rangle = -2q_1^2|\Lambda^{(s+1)k}_{00}\rangle + \eta^+|\Lambda^{(s+1)k}_{012}\rangle\,, \tag{6.7}$$
$$\delta|\Lambda^{(s)k}_{01}\rangle = -2q_1^2|\Lambda^{(s+1)k}_{012}\rangle\,, \tag{6.8}$$
$$\delta|\Lambda^{(s)k}_{02}\rangle = -2q_1^2|\Lambda^{(s+1)k}_{022}\rangle + \eta^+|\Lambda^{(s+1)k}_{04}\rangle\,, \tag{6.9}$$
$$\delta|\Lambda^{(s)k}_{03}\rangle = -2q_1^2|\Lambda^{(s+1)k}_{04}\rangle\,. \tag{6.10}$$

Then, a certain choice for $|\Lambda^{(s+1)k}_{04}\rangle$, $|\Lambda^{(s+1)k}_{012}\rangle$ removes $|\Lambda^{(s)k}_{03}\rangle$, $|\Lambda^{(s)k}_{01}\rangle$, respectively, whereas a certain choice for $|\Lambda^{(s+1)k}_{022}\rangle$, $|\Lambda^{(s+1)k}_{00}\rangle$ eliminates $|\Lambda^{(s)k}_{02}\rangle$, $|\Lambda^{(s)k}_{011}\rangle$ by means of the remaining gauge transformations. Thus, we have shown that the dependence on $\mathcal{P}^+_{11}$ can be eliminated from $|\Lambda^{(l)k}_0\rangle$. As a consequence of the above procedure, the theory becomes a gauge theory of l-th stage of reducibility.

This algorithm is valid down to the vector $|\Lambda^{(n_2+1)k}_0\rangle$, when there arise terms linear in p^+_1. When these terms are present, one deals with gauge parameters that have remained unused after eliminating the dependence on $\mathcal{P}^+_{11}$. Therefore, in view of the η^+-dependent terms in (6.7), (6.9), a gauge transformation with such parameters may cause some $\mathcal{P}^+_{11}$-dependent terms to appear in the transformed vector $|\Lambda^{(n_2)k}_0\rangle$. Consequently, it is necessary to make a gauge transformation with parameters linear in p^+_1, or independent of it, before removing the $\mathcal{P}^+_{11}$-dependence. Let us examine a gauge transformation with the gauge function $|\Lambda^{(n_2)k}_0\rangle$ more carefully.

Suppose that the dependence on the ghost $\mathcal{P}^+_{11}$ in $|\Lambda^{(n_2+1)k}_0\rangle$ has been removed by a gauge transformation, and hence the functions $|\Lambda^{(n_2+1)k}_0\rangle$, $|\Lambda^{(n_2)k}_0\rangle$ admit the following representation:

$$|\Lambda^{(n_2+1)0}_0\rangle = p^+_1\Big(|\Lambda^{(n_2+1)0}_{00}\rangle + \mathcal{P}^+_{12}|\Lambda^{(n_2+1)0}_{012}\rangle + p^+_1\mathcal{P}^+_{22}|\Lambda^{(n_2+1)0}_{022}\Big)\rangle\,, \tag{6.11}$$
$$|\Lambda^{(n_2+1)1}_0\rangle = (p^+_1)^2\Big(|\Lambda^{(n_2+1)1}_{01}\rangle + \mathcal{P}^+_{12}|\Lambda^{(n_2+1)1}_{012}\rangle + p^+_1\mathcal{P}^+_{22}|\Lambda^{(n_2+1)1}_{022}\rangle\Big)\,, \tag{6.12}$$
$$|\Lambda^{(n_2)0}_0\rangle = |\Lambda^{(n_2)0}_{00}\rangle + \sum_{i\leq j}\mathcal{P}^+_{ij}|\Lambda^{(n_2)0}_{0ij}\rangle\,, \tag{6.13}$$
$$|\Lambda^{(n_2)1}_0\rangle = p^+_1\Big(|\tilde{\Lambda}^{(n_2)1}_{00}\rangle + \mathcal{P}^+_{12}|\tilde{\Lambda}^{(n_2)1}_{012}\rangle + p^+_1\mathcal{P}^+_{22}|\tilde{\Lambda}^{(n_2)1}_{022}\rangle\Big)\,, \tag{6.14}$$

where the vectors $|\Lambda^{(l)}{}^{0}_{00}\rangle$, $|\Lambda^{(l)k}_{0ij}\rangle$, $l = n_2, n_2+1$, $i,j = 1,2, i \leq j$, possess terms having no dependence on p_1^+, except for $|\Lambda^{(n_2)0}_{022}\rangle$, and the vectors in (6.14) have a structure analogous to the corresponding structure in (6.11). Then, one has to make a transformation with parameters linear in p_1^+. We will use $|\Lambda^{(n_2+1)}{}^{0}_{00}\rangle$, $|\Lambda^{(n_2+1)0}_{012}\rangle$ to make such gauge transformations. Since

$$\delta|\Lambda^{(n_2)}{}^{0}_{00}\rangle = T_1^+|\hat{\Lambda}^{(n_2+1)}{}^{0}_{00}\rangle\,, \qquad |\hat{\Lambda}^{(n_2+1)}{}^{0}_{00}\rangle = q_1|\Lambda^{(n_2+1)}{}^{0}_{00}\rangle\,, \tag{6.15}$$

$$\delta|\Lambda^{(n_2)0}_{012}\rangle = T_1^+|\hat{\Lambda}^{(n_2+1)0}_{012}\rangle\,, \qquad |\hat{\Lambda}^{(n_2+1)0}_{012}\rangle = q_1|\Lambda^{(n_2+1)0}_{012}\rangle\,, \tag{6.16}$$

one can use the vectors $|\Lambda^{(n_2+1)}{}^{0}_{00}\rangle$, $|\Lambda^{(n_2+1)0}_{012}\rangle$ to eliminate the dependence on b_{11}^+ and f_1^+ from $|\Lambda^{(n_2)}{}^{0}_{00}\rangle$ and $|\Lambda^{(n_2)0}_{012}\rangle$, respectively, due to the fact that the b_{11}^+- and f_1^+-linear components of the latter vectors are identical to the corresponding components of the previous vectors. As a result, we obtain the gauge-fixing

$$b_{11}|\Lambda^{(n_2)}{}^{0}_{00}\rangle = f_1|\Lambda^{(n_2)}{}^{0}_{00}\rangle = b_{11}|\Lambda^{(n_2)0}_{012}\rangle = f_1|\Lambda^{(n_2)0}_{012}\rangle = 0\,, \tag{6.17}$$

and then remove the $\mathcal{P}_{11}^+$-dependence from $|\Lambda^{(n_2)}{}^{j}_{0}\rangle$, as has been described in the case of the system (6.7)–(6.10).

Proceeding by induction, we may use the algorithm which has been applied to the treatment of the vectors (6.13), (6.14) in order to eliminate the dependence on $\mathcal{P}_{11}^+$ related to all the vectors down to $|\Lambda^{(0)}{}^{j}_{0}\rangle$, whereas the $\mathcal{P}_{22}^+$-independent terms in $|\Lambda^{(l)}{}^{j}_{0}\rangle$ for $l \geq n_2$ are restricted by relations of the form (6.17).

Let us now turn to the gauge-fixing of the fields. We expand the fields in the powers of the ghosts $\mathcal{P}_{ij}^+$ by analogy with the gauge parameters:

$$|\chi_0^k\rangle = |\chi^k_{00}\rangle + \sum_{i\leq j}\mathcal{P}^+_{ij}|\chi^k_{0ij}\rangle + \mathcal{P}^+_{11}\Big(\mathcal{P}^+_{12}|\chi^k_{01}\rangle + \mathcal{P}^+_{22}|\chi^k_{02}\rangle + \mathcal{P}^+_{12}\mathcal{P}^+_{22}|\chi^k_{03}\rangle\Big) + \mathcal{P}^+_{12}\mathcal{P}^+_{22}|\chi^k_{04}\rangle\,. \tag{6.18}$$

Further, we need to restrict the vectors by the gauge conditions (6.17), which follow from the gauge transformations, and then we eliminate the terms coupled to $\mathcal{P}_{11}^+$.

Having completed the above procedure, we briefly mention that the remaining gauge ambiguity is sufficient to eliminate the auxiliary oscillators b_{ij}^+, b^+, f_i^+ from the field $|\chi^{00000}_{00}\rangle$,

$$|\chi^k_{00}\rangle = \sum_{l_i\geq 0}\sum_{m_j=0}^{1}(p_i^+)^{l_i}(\mathcal{P}_j^+)^{m_j}|\chi^{kl_im_j}_{00}\rangle\,, \tag{6.19}$$

and therefore, in view of $gh(|\chi^{00000}_{00}\rangle) = 0$, this field has no dependence on the ghost "coordinates", so that, after the gauge-fixing, we conclude

$$|\chi^{00000}_{00}\rangle = |\Phi\rangle\,. \tag{6.20}$$

The second step of establishing an equivalence of equations (2.2)–(2.5) with the Lagrangian equations (5.24), (5.25) is more involved and is based on a detailed expansion of

equations (5.24), (5.25) in the powers of p_i^+, q_i^+, η_i^+, η_{ij}^+, η^+ and then in the powers of b_{ij}^+, f_i^+. We only state the result that after gauge-fixing $|\chi^k_{00}\rangle$ and $|\chi^k_{012}\rangle$ $|\chi^k_{022}\rangle$, $|\chi^k_{04}\rangle$, expanded by analogy with (6.19), the only independent equations among (5.24), (5.25) have the form

$$t_0|\Phi\rangle = t_i|\Phi\rangle = t|\Phi\rangle = 0\,, \tag{6.21}$$

and all of the auxiliary fields can be made equal to zero.

In what follows, we consider some examples of the Lagrangian formulation procedure.

7. Examples

Here, we shall realize the general prescriptions of our Lagrangian formulation in the case of fermionic fields of lowest spins.

7.1 Spin-(3/2,1/2) field

In the case of a field of spin (3/2,1/2), we have $(n_1, n_2) = (1, 0)$, $(h^1, h^2) = (1-d/2, 4-d/2)$. Since $s_{\max} = 0$, the corresponding Lagrangian formulation is an irreducible gauge theory and describes a totally symmetric fermionic field of spin $s = 3/2$. The nonvanishing fields $|\chi^i_0\rangle_{(1,1)}$ and gauge parameters $|\Lambda^0_0\rangle_{(1,0)}$, (for $|\Lambda^1_0\rangle_{(1,0)} \equiv 0$, due to $gh(|\Lambda^1_0\rangle_{(1,0)}) = -2$), have the following Grassmann grading and ghost number distribution:

$$(\varepsilon, gh)\,(|\chi^i_0\rangle_{(1,0)}) = (1, -i)\,, \qquad (\varepsilon, gh)\,(|\Lambda^0_0\rangle_{(1,0)}) = (0, -1)\,. \tag{7.1}$$

These conditions determine the dependence of the fields and gauge parameters on the oscillator variables in a unique form, with the help of the operators corresponding only to the first row of the Young tableaux,

$$|\chi^0_0\rangle_{(1,0)} = \Big[-ia_1^{+\mu}\psi_\mu(x) + f_1^+\tilde\gamma\psi(x)\Big]\,|0\rangle, \quad |\chi^1_0\rangle_{(1,0)} = \big[\mathcal{P}_1^+\tilde\gamma\chi(x) + p_1^+\chi_1(x)\big]\,|0\rangle, \tag{7.2}$$

$${}_{(1,0)}\langle\tilde\chi^0_0| = \langle 0|\left[i\psi^+_\mu(x)a_1^\mu + \psi^+(x)\tilde\gamma f_1\right]\tilde\gamma^0, \quad {}_{(1,0)}\langle\tilde\chi^1_0| = \langle 0|\left[\chi^+(x)\tilde\gamma\mathcal{P}_1 + \chi_1^+ p_1\right]\tilde\gamma^0, \tag{7.3}$$

$$|\Lambda^0_0\rangle_{(1,0)} = \big[\mathcal{P}_1^+\xi_1(x) + p_1^+\tilde\gamma\xi_2(x)\big]\,|0\rangle, \tag{7.4}$$

Substituting (7.2), (7.3) into (5.27), we find the action (up to an overall factor) for a free massless field of spin $(3/2, 1/2)$ on a flat background:

$$\begin{aligned}\mathcal{S}_{(1,0)} = \int d^dx\,\Big[&\bar\psi^\mu\Big\{i\gamma^\nu\partial_\nu\psi_\mu - \partial_\mu\chi\Big\} + (d-2)\bar\psi\Big\{i\gamma^\mu\partial_\mu\psi + \chi_1\Big\} \\ &+\bar\chi\Big\{i\gamma^\mu\partial_\mu\chi - \chi_1 + \partial^\mu\psi_\mu\Big\} + \bar\chi_1\Big\{i\gamma^\mu\psi_\mu + (d-2)\psi - \chi\Big\}\Big]\,.\end{aligned} \tag{7.5}$$

In deriving the action (7.5), we have used the expressions (3.36), (4.8) for the operators $K_{(1,0)}$.[10] A substitution of (7.2)–(7.4) into (5.28), (5.29) permits one to find the gauge

[10] For $n_2 = 0$, we have the case of totally symmetric spin-tensors in a d-dimensional flat space [44], so that the total Hilbert space $\mathcal{H}_{\rm tot}$ and all of the operators acting on it can be factorized from $q_2^+, \eta_2^+, \eta_{12}^+, \eta_{22}^+, \eta^+$, $q_2, \eta_2, \eta_{12}, \eta_{22}, \eta$, $f_2^+, b_{12}^+, b_{12}^+, b^+$ and their canonically conjugate operators. In the expressions for the action (5.27) and the sequence of gauge transformations (5.28)–(5.31), we must set $n_2 = 0$ and use the above restrictions for $\mathcal{H}_{\rm tot}$. In particular, the operator $K'_{(n_1,0)}$ has an exact form [44], $K'_{(n_1,0)} = \sum_{n_{11}=0}\frac{1}{n_{11}!}\Big(\,|n_{11}\rangle\langle n_{11}|\,C(n_{11}, h_{n_1}) - 2f_1^+|n_{11}\rangle\langle n_{11}|f_1\,C(n_{11}+1, h_{n_1})\,\Big)$, for $C(n,h) = h(h+1)\cdots(h+n-1), C(0,h) = 1$.

transformations (5.28), (5.29) in the form

$$\delta\psi_\mu = \partial_\mu\xi_1 + i\gamma_\mu\xi_2, \quad \delta\psi = \xi_2, \quad \delta\chi = i\gamma^\mu\partial_\mu\xi_1 - 2\xi_2, \quad \delta\chi_1 = -i\gamma^\mu\partial_\mu\xi_2. \tag{7.6}$$

Let us present the action in terms of the physical field ψ_μ alone. To this end, we get rid of the field ψ by using its gauge transformation and the gauge parameter ξ_2. Having expressed the field χ by using the equation of motion $\chi = i\gamma^\mu\psi_\mu$, we can see that the terms with the Lagrangian multiplier χ_1 turn to zero. As a result, we obtain the action

$$\mathcal{S}_{(1,0)} = \int d^dx\Big\{i\bar\psi^\mu\gamma^\nu\partial_\nu\psi_\mu - i\bar\psi^\mu(\gamma_\nu\partial_\mu + \gamma_\mu\partial_\nu)\psi^\nu + i\bar\psi^\nu\gamma_\nu\gamma^\sigma\partial_\sigma\gamma^\mu\psi_\mu\Big\}, \tag{7.7}$$

which is invariant with respect to the residual gauge transformation $\delta\psi_\mu = \partial_\mu\xi_1$.

To obtain a Lagrangian description of a massive fermionic field, we may use the prescription (5.34)–(5.36) either from the very beginning of our treatment of this example, i.e., starting from expansion (7.2), (7.3), or starting directly from the action (7.7) written for a $(d+1)$-dimensional Minkowski space. Following the latter possibility, for a spin-3/2 massive fermionic field, with $\psi^M = (\psi^\mu, \varphi)$, the Stueckelberg field being φ, and with $i\gamma^M\partial_M\psi^N = (i\gamma^\mu\partial_\mu - m)\psi^N$, $i\gamma^M\partial_M\gamma_N\psi^N = (i\gamma^\mu\partial_\mu + m)\gamma_N\psi^N$ in view of (5.36), and due to the relation $\chi = i\gamma_N\psi^N$ before eq. (7.7) and the structure of $|\chi_0^1\rangle_{(1,0)}$ in (7.2), the Lagrangian reads

$$\begin{aligned}\mathcal{L}^m_{(1,0)} = {}& \bar\psi^\mu(i\gamma^\nu\partial_\nu - m)\psi_\mu - \bar\varphi(i\gamma^\nu\partial_\nu - m)\varphi - i\bar\psi^\mu(\gamma_\nu\partial_\mu + \gamma_\mu\partial_\nu)\psi^\nu \\ & + i\bar\psi^\mu(\partial_\mu - im\gamma_\mu)\varphi + i\bar\varphi(\partial_\mu - im\gamma_\mu)\psi^\mu + 2\bar\varphi m\varphi \\ & + (\bar\psi^\nu\gamma_\nu + \bar\varphi)(i\gamma^\sigma\partial_\sigma + m)(\gamma^\mu\psi_\mu + \varphi),\end{aligned} \tag{7.8}$$

and is invariant, modulo a total derivative, with respect to the gauge transformation

$$\delta\psi_\mu = \partial_\mu\xi, \qquad \delta\varphi = m\xi. \tag{7.9}$$

After removing the field φ by means of the gauge transformations, we obtain the Rarita-Schwinger Lagrangian in a d-dimensional flat space [44], indeed,

$$\mathcal{L}_{RS} = \bar\psi^\mu(i\gamma^\nu\partial_\nu - m)\psi_\mu - i\bar\psi^\mu(\gamma^\nu\partial_\mu + \gamma_\mu\partial^\nu)\psi_\nu + \bar\psi_\mu\gamma^\mu(i\gamma^\sigma\partial_\sigma + m)\gamma^\nu\psi_\nu. \tag{7.10}$$

7.2 Rank-2 antisymmetric spin-tensor field

In the case of a spin-(3/2,3/2) field, we have $n_i = 1$, $(h^1, h^2) = (1 - d/2, 3 - d/2)$. Since $s_{\max} = 2$, the corresponding Lagrangian description is a reducible gauge theory of second-stage reducibility. The nonvanishing fields $|\chi_0^0\rangle_{(1,1)}$, $|\chi_0^1\rangle_{(1,1)}$, gauge parameters $|\Lambda_0^k\rangle_{(1,1)}$, first-stage gauge parameters $|\Lambda_0^{(1)k}\rangle_{(1,1)}$, and second-stage gauge parameters (for $|\Lambda_0^{(2)1}\rangle_{(1,1)} \equiv 0$, due to $gh(|\Lambda_0^{(2)1}\rangle_{(1,1)}) = -4$), have the following Grassmann grading and ghost number distribution:

$$(\varepsilon, gh)\,(|\chi_0^i\rangle_{(1,1)}) = (1, -i)\,, \qquad (\varepsilon, gh)\,(|\Lambda_0^k\rangle_{(1,1)}) = (0, -1-k)\,, \tag{7.11}$$

$$(\varepsilon, gh)\,(|\Lambda_0^{(1)k}\rangle_{(1,1)}) = (1, -2-k)\,, \qquad (\varepsilon, gh)\,(|\Lambda_0^{(2)0}\rangle_{(1,1)}) = (0, -3)\,. \tag{7.12}$$

These conditions allow one, first, to extract the dependence on the ghost variables from the fields and gauge parameters:

$$
\begin{aligned}
|\chi_0^0\rangle_{(1,1)} = {} & |\Psi\rangle_{(1,1)} + \eta_1^+\mathcal{P}_2^+|\Psi_1\rangle_{(0,0)} + \mathcal{P}_1^+\eta_2^+|\Psi_2\rangle_{(0,0)} + q_1^+p_2^+|\Psi_3\rangle_{(0,0)} \\
& + p_1^+q_2^+|\Psi_4\rangle_{(0,0)} + \eta_1^+p_2^+\tilde{\gamma}|\Psi_5\rangle_{(0,0)} + q_1^+\mathcal{P}_2^+\tilde{\gamma}|\Psi_6\rangle_{(0,0)} + \mathcal{P}_1^+q_2^+\tilde{\gamma}|\Psi_7\rangle_{(0,0)} \\
& + p_1^+\eta_2^+\tilde{\gamma}|\Psi_8\rangle_{(0,0)} + \mathcal{P}_{11}^+\eta^+|\Psi_9\rangle_{(0,0)} + \eta_{11}^+\mathcal{P}^+|\Psi_{10}\rangle_{(0,0)} \\
& + q_1^+\mathcal{P}^+\tilde{\gamma}|\varphi_1\rangle_{(1,0)} + \eta_1^+\mathcal{P}^+|\varphi_2\rangle_{(1,0)} + \mathcal{P}_1^+\eta^+|\varphi_3\rangle_{(1,0)} + p_1^+\eta^+\tilde{\gamma}|\varphi_4\rangle_{(1,0)} \\
& + q_1^+p_1^+|\rho_1\rangle_{(-1,1)} + q_1^+\mathcal{P}_1^+\tilde{\gamma}|\rho_2\rangle_{(-1,1)} + \eta_1^+p_1^+\tilde{\gamma}|\rho_3\rangle_{(-1,1)} + \eta_1^+\mathcal{P}_1^+|\rho_4\rangle_{(-1,1)}\,,
\end{aligned}
\tag{7.13}
$$

$$
\begin{aligned}
|\chi_0^1\rangle_{(1,1)} = {} & p_1^+|\chi_1\rangle_{(0,1)} + p_2^+|\chi_2\rangle_{(1,0)} + \mathcal{P}_1^+\tilde{\gamma}|\chi_3\rangle_{(0,1)} + \mathcal{P}_2^+\tilde{\gamma}|\chi_4\rangle_{(1,0)} \\
& + \mathcal{P}_{12}^+\tilde{\gamma}|\chi_5\rangle_{(0,0)} + q_1^+p_1^+\mathcal{P}^+\tilde{\gamma}|\chi_6\rangle_{(0,0)} + \eta_1^+\mathcal{P}_1^+\mathcal{P}^+\tilde{\gamma}|\chi_7\rangle_{(0,0)} \\
& + q_1^+\mathcal{P}_1^+\mathcal{P}^+|\chi_8\rangle_{(0,0)} + p_1^+\eta_1^+\mathcal{P}^+|\chi_9\rangle_{(0,0)} + p_1^+\mathcal{P}_1^+\eta^+|\chi_{10}\rangle_{(0,0)} \\
& + (p_1^+)^2\eta^+\tilde{\gamma}|\chi_{11}\rangle_{(0,0)} + \mathcal{P}^+\tilde{\gamma}|\chi_{12}\rangle_{(2,0)} + \mathcal{P}_{11}^+\tilde{\gamma}|\chi_{13}\rangle_{(-1,1)}\,,
\end{aligned}
\tag{7.14}
$$

$$
\begin{aligned}
|\Lambda_0^0\rangle_{(1,1)} = {} & p_1^+\tilde{\gamma}|\xi_1\rangle_{(0,1)} + p_2^+\tilde{\gamma}|\xi_2\rangle_{(1,0)} + \mathcal{P}_1^+|\xi_3\rangle_{(0,1)} + \mathcal{P}_2^+|\xi_4\rangle_{(1,0)} \\
& + \mathcal{P}_{12}^+|\xi_5\rangle_{(0,0)} + q_1^+p_1^+\mathcal{P}^+|\xi_6\rangle_{(0,0)} + \eta_1^+\mathcal{P}_1^+\mathcal{P}^+|\xi_7\rangle_{(0,0)} \\
& + q_1^+\mathcal{P}_1^+\mathcal{P}^+\tilde{\gamma}|\xi_8\rangle_{(0,0)} + p_1^+\eta_1^+\mathcal{P}^+\tilde{\gamma}|\xi_9\rangle_{(0,0)} + p_1^+\mathcal{P}_1^+\eta^+\tilde{\gamma}|\xi_{10}\rangle_{(0,0)} \\
& + (p_1^+)^2\eta^+|\xi_{11}\rangle_{(0,0)} + \mathcal{P}^+|\xi_{12}\rangle_{(2,0)} + \mathcal{P}_{11}^+|\xi_{13}\rangle_{(-1,1)}\,,
\end{aligned}
\tag{7.15}
$$

$$
\begin{aligned}
|\Lambda_0^1\rangle_{(1,1)} = {} & p_1^+p_2^+\tilde{\gamma}|\lambda_1\rangle_{(0,0)} + p_1^+\mathcal{P}_2^+|\lambda_2\rangle_{(0,0)} + \mathcal{P}_1^+p_2^+|\lambda_3\rangle_{(0,0)} \\
& + \mathcal{P}_1^+\mathcal{P}_2^+\tilde{\gamma}|\lambda_4\rangle_{(0,0)} + \mathcal{P}_{11}^+\mathcal{P}^+\tilde{\gamma}|\lambda_5\rangle_{(0,0)} + \mathcal{P}_1^+\mathcal{P}^+\tilde{\gamma}|\lambda_6\rangle_{(1,0)} \\
& + p_1^+\mathcal{P}^+|\lambda_7\rangle_{(1,0)} + (p_1^+)^2\tilde{\gamma}|\lambda_8\rangle_{(-1,1)} + p_1^+\mathcal{P}_1^+|\lambda_9\rangle_{(-1,1)}\,,
\end{aligned}
\tag{7.16}
$$

$$
\begin{aligned}
|\Lambda_0^{(1)0}\rangle_{(1,1)} = {} & p_1^+p_2^+|\xi_1^{(1)}\rangle_{(0,0)} + p_1^+\mathcal{P}_2^+\tilde{\gamma}|\xi_2^{(1)}\rangle_{(0,0)} + \mathcal{P}_1^+p_2^+\tilde{\gamma}|\xi_3^{(1)}\rangle_{(0,0)} \\
& + \mathcal{P}_1^+\mathcal{P}_2^+|\xi_4^{(1)}\rangle_{(0,0)} + \mathcal{P}_{11}^+\mathcal{P}^+|\xi_5^{(1)}\rangle_{(0,0)} + \mathcal{P}_1^+\mathcal{P}^+|\xi_6^{(1)}\rangle_{(1,0)} \\
& + p_1^+\mathcal{P}^+\tilde{\gamma}|\xi_7^{(1)}\rangle_{(1,0)} + (p_1^+)^2|\xi_8^{(1)}\rangle_{(-1,1)} + p_1^+\mathcal{P}_1^+\tilde{\gamma}|\xi_9^{(1)}\rangle_{(-1,1)}\,,
\end{aligned}
\tag{7.17}
$$

$$
|\Lambda_0^{(1)1}\rangle_{(1,1)} = (p_1^+)^2\mathcal{P}^+\tilde{\gamma}|\lambda_1^{(1)}\rangle_{(0,0)} + p_1^+\mathcal{P}_1^+\mathcal{P}^+|\lambda_2^{(1)}\rangle_{(0,0)}\,, \tag{7.18}
$$

$$
|\Lambda_0^{(2)0}\rangle_{(1,1)} = (p_1^+)^2\mathcal{P}^+|\xi_1^{(2)}\rangle_{(0,0)} + p_1^+\mathcal{P}_1^+\mathcal{P}^+\tilde{\gamma}|\xi_2^{(2)}\rangle_{(0,0)}\,, \tag{7.19}
$$

where the coefficient fermionic fields and gauge parameters in the right-hand side of equations (7.13)–(7.19) are independent of ghost operators. The bra-vectors ${}_{(1,1)}\langle\tilde{\chi}_0^k|$ corresponding to expansion (7.13), (7.14) have the form

$$
\begin{aligned}
{}_{(1,1)}\langle\tilde{\chi}_0^0| = {} & {}_{(1,1)}\langle\tilde{\Psi}| + {}_{(0,0)}\langle\tilde{\Psi}_1|\mathcal{P}_2\eta_1 + {}_{(0,0)}\langle\tilde{\Psi}_2|\eta_2\mathcal{P}_1 + {}_{(0,0)}\langle\tilde{\Psi}_3|q_1p_2 \\
& + {}_{(0,0)}\langle\tilde{\Psi}_4|q_2p_1 + {}_{(0,0)}\langle\tilde{\Psi}_5|\tilde{\gamma}p_2\eta_1 + {}_{(0,0)}\langle\tilde{\Psi}_6|\tilde{\gamma}\mathcal{P}_2q_1 + {}_{(0,0)}\langle\tilde{\Psi}_7|\tilde{\gamma}q_2\mathcal{P}_1 \\
& + {}_{(0,0)}\langle\tilde{\Psi}_8|\tilde{\gamma}\eta_2p_1 + {}_{(0,0)}\langle\tilde{\Psi}_9|\eta\mathcal{P}_{11} + {}_{(0,0)}\langle\tilde{\Psi}_{10}|\mathcal{P}\eta_{11} \\
& + {}_{(1,0)}\langle\tilde{\varphi}_1|\tilde{\gamma}\mathcal{P}q_1 + {}_{(1,0)}\langle\tilde{\varphi}_2|\mathcal{P}\eta_1 + {}_{(1,0)}\langle\tilde{\varphi}_3|\eta\mathcal{P}_1 + {}_{(1,0)}\langle\tilde{\varphi}_4|\tilde{\gamma}\eta p_1 \\
& + {}_{(-1,1)}\langle\tilde{\rho}_1|p_1q_1 + {}_{(-1,1)}\langle\tilde{\rho}_2|\tilde{\gamma}\mathcal{P}_1q_1 + {}_{(-1,1)}\langle\tilde{\rho}_3|\tilde{\gamma}p_1\eta_1 + {}_{(-1,1)}\langle\tilde{\rho}_4|\mathcal{P}_1\eta_1\,,
\end{aligned}
\tag{7.20}
$$

$$
\begin{aligned}
{}_{(1,1)}\langle\tilde{\chi}_0^1| = {} & {}_{(0,1)}\langle\tilde{\chi}_1|p_1 + {}_{(1,0)}\langle\tilde{\chi}_2|p_2 + {}_{(0,1)}\langle\tilde{\chi}_3|\tilde{\gamma}\mathcal{P}_1 + {}_{(1,0)}\langle\tilde{\chi}_4|\tilde{\gamma}\mathcal{P}_2 \\
& + {}_{(0,0)}\langle\tilde{\chi}_5|\tilde{\gamma}\mathcal{P}_{12} + {}_{(0,0)}\langle\tilde{\chi}_6|\tilde{\gamma}\mathcal{P}p_1q_1 + {}_{(0,0)}\langle\tilde{\chi}_7|\tilde{\gamma}\mathcal{P}\mathcal{P}_1\eta_1 \\
& + {}_{(0,0)}\langle\tilde{\chi}_8|\mathcal{P}\mathcal{P}_1q_1 + {}_{(0,0)}\langle\tilde{\chi}_9|\mathcal{P}\eta_1p_1 + {}_{(0,0)}\langle\tilde{\chi}_{10}|\eta\mathcal{P}_1p_1 \\
& + {}_{(0,0)}\langle\tilde{\chi}_{11}|\tilde{\gamma}\eta p_1^2 + {}_{(2,0)}\langle\tilde{\chi}_{12}|\tilde{\gamma}\mathcal{P} + {}_{(-1,1)}\langle\tilde{\chi}_{13}|\tilde{\gamma}\mathcal{P}_{11}\,.
\end{aligned}
\tag{7.21}
$$

Substituting (7.13), (7.14), (7.20), (7.21) into (5.27), we find the action (up to an overall factor) for a spin-$(3/2, 3/2)$ free massless field on a flat background in the form of a scalar product for vectors defined only in $\mathcal{H} \otimes \mathcal{H}'$,

$$
\begin{aligned}
\mathcal{S}_{(1,1)} = \Bigg[& \langle \tilde{\Psi} | K_{(1,1)} \left\{ \frac{1}{2} T_0 |\Psi\rangle + T_1^+ |\chi_1\rangle + T_2^+ |\chi_2\rangle + \tilde{\gamma} L_1^+ |\chi_3\rangle \right. \\
& \left. + \tilde{\gamma} L_2^+ |\chi_4\rangle + \tilde{\gamma} L_{12}^+ |\chi_5\rangle + \tilde{\gamma} T^+ |\chi_{12}\rangle + \tilde{\gamma} L_{11}^+ |\chi_{13}\rangle \right\} \\
& + \langle \tilde{\Psi}_1 | K_{(1,1)} \left\{ T_0 |\Psi_2\rangle - 2\tilde{\gamma} |\Psi_8\rangle - \tilde{\gamma} L_2 |\chi_3\rangle + \frac{1}{2} \tilde{\gamma} |\chi_5\rangle + \tilde{\gamma} |\chi_7\rangle \right\} \\
& + \langle \tilde{\Psi}_2 | K_{(1,1)} \left\{ 2\tilde{\gamma} |\Psi_5\rangle + \tilde{\gamma} L_1 |\chi_4\rangle - \frac{1}{2} \tilde{\gamma} |\chi_5\rangle - \tilde{\gamma} |\chi_7\rangle \right\} \\
& + \langle \tilde{\Psi}_3 | K_{(1,1)} \left\{ T_0 |\Psi_4\rangle - 2\tilde{\gamma} |\Psi_8\rangle + T_2 |\chi_1\rangle + \frac{1}{2} \tilde{\gamma} |\chi_5\rangle - \tilde{\gamma} |\chi_6\rangle - 4\tilde{\gamma} |\chi_{11}\rangle \right\} \\
& + \langle \tilde{\Psi}_4 | K_{(1,1)} \left\{ -2\tilde{\gamma} |\Psi_5\rangle + T_1 |\chi_2\rangle + \frac{1}{2} \tilde{\gamma} |\chi_5\rangle - \tilde{\gamma} |\chi_6\rangle \right\} \\
& + \langle \tilde{\Psi}_5 | K_{(1,1)} \left\{ -T_0 |\Psi_7\rangle - T_2 |\chi_3\rangle + \tilde{\gamma} |\chi_8\rangle + 2\tilde{\gamma} |\chi_{10}\rangle \right\} \\
& + \langle \tilde{\Psi}_6 | K_{(1,1)} \left\{ -T_0 |\Psi_8\rangle + \tilde{\gamma} L_2 |\chi_1\rangle + \tilde{\gamma} |\chi_9\rangle \right\} \\
& + \langle \tilde{\Psi}_7 | K_{(1,1)} \left\{ \tilde{\gamma} L_1 |\chi_2\rangle + \tilde{\gamma} |\chi_9\rangle \right\} + \langle \tilde{\Psi}_8 | K_{(1,1)} \left\{ -T_1 |\chi_4\rangle + \tilde{\gamma} |\chi_8\rangle \right\} \\
& + \langle \tilde{\Psi}_9 | K_{(1,1)} \left\{ T_0 |\Psi_{10}\rangle - \frac{1}{2} \tilde{\gamma} |\chi_5\rangle + \tilde{\gamma} |\chi_6\rangle + \tilde{\gamma} |\chi_7\rangle + \tilde{\gamma} L_{11} |\chi_{12}\rangle \right\} \\
& + \langle \tilde{\Psi}_{10} | K_{(1,1)} \left\{ \tilde{\gamma} |\chi_5\rangle - 4\tilde{\gamma} |\chi_{11}\rangle - \tilde{\gamma} T |\chi_{13}\rangle \right\} \\
& + \langle \tilde{\varphi}_1 | K_{(1,1)} \left\{ -T_0 |\varphi_4\rangle + \tilde{\gamma} T |\chi_1\rangle - \tilde{\gamma} |\chi_2\rangle + \tilde{\gamma} L_1^+ |\chi_{10}\rangle - 2T_1^+ |\chi_{11}\rangle \right\} \\
& + \langle \tilde{\varphi}_2 | K_{(1,1)} \left\{ T_0 |\varphi_3\rangle - 2\tilde{\gamma} |\varphi_4\rangle - \tilde{\gamma} T |\chi_3\rangle + \tilde{\gamma} |\chi_4\rangle + T_1^+ |\chi_{10}\rangle \right\} \\
& + \langle \tilde{\varphi}_3 | K_{(1,1)} \left\{ -\tilde{\gamma} L_1^+ |\chi_7\rangle + T_1^+ |\chi_9\rangle + \tilde{\gamma} L_1 |\chi_{12}\rangle \right\} \\
& + \langle \tilde{\varphi}_4 | K_{(1,1)} \left\{ -2\tilde{\gamma} |\chi_2\rangle - T_1^+ |\chi_6\rangle - T_1 |\chi_{12}\rangle \right\} \\
& + \langle \tilde{\rho}_1 | K_{(1,1)} \left\{ \frac{1}{2} T_0 |\rho_1\rangle - 2\tilde{\gamma} |\rho_3\rangle + T_1 |\chi_1\rangle + \tilde{\gamma} T^+ |\chi_6\rangle + \tilde{\gamma} |\chi_{13}\rangle \right\} \\
& + \langle \tilde{\rho}_2 | K_{(1,1)} \left\{ -T_0 |\rho_3\rangle + \tilde{\gamma} L_1 |\chi_1\rangle - \tilde{\gamma} T^+ |\chi_9\rangle \right\} \\
& + \langle \tilde{\rho}_3 | K_{(1,1)} \left\{ -2\tilde{\gamma} |\rho_4\rangle - T_1 |\chi_3\rangle - \tilde{\gamma} T^+ |\chi_8\rangle \right\} \\
& + \langle \tilde{\rho}_4 | K_{(1,1)} \left\{ -\frac{1}{2} T_0 |\rho_4\rangle - \tilde{\gamma} L_1 |\chi_3\rangle - \tilde{\gamma} T^+ |\chi_7\rangle + \tilde{\gamma} |\chi_{13}\rangle \right\} \\
& + \langle \tilde{\chi}_1 | K_{(1,1)} \left\{ -\tilde{\gamma} |\chi_3\rangle \right\} + \langle \tilde{\chi}_2 | K_{(1,1)} \left\{ -\tilde{\gamma} |\chi_4\rangle \right\} + \langle \tilde{\chi}_3 | K_{(1,1)} \left\{ -\frac{1}{2} T_0 |\chi_3\rangle \right\} \\
& + \langle \tilde{\chi}_4 | K_{(1,1)} \left\{ -\frac{1}{2} T_0 |\chi_4\rangle \right\} + \langle \tilde{\chi}_6 | K_{(1,1)} \left\{ -\tilde{\gamma} |\chi_{10}\rangle \right\} + \langle \tilde{\chi}_7 | K_{(1,1)} \left\{ -\tilde{\gamma} |\chi_{10}\rangle \right\} \\
& + \langle \tilde{\chi}_8 | K_{(1,1)} \left\{ T_0 |\chi_{10}\rangle - 2\tilde{\gamma} |\chi_{11}\rangle \right\} \Bigg] + c.c. \,, \qquad (7.22)
\end{aligned}
$$

where we have omitted the lower spin subscripts of the component fields. In deriving the action (7.22), we have used the expressions for the operators $K_{(1,1)}$ (3.36), (4.8), and then, substituting (7.2)–(7.4) into (5.28), (5.29), we find the gauge transformations for the

vectors $|\Psi\rangle, |\Psi_k\rangle$,

$$\delta|\Psi\rangle = -\tilde{\gamma}(T_1^+|\xi_1\rangle+T_2^+|\xi_2\rangle)+L_1^+|\xi_3\rangle+L_2^+|\xi_4\rangle+L_{12}^+|\xi_5\rangle+L_{11}^+|\xi_{13}\rangle+T^+|\xi_{12}\rangle, \quad (7.23)$$

$$\delta|\Psi_1\rangle = L_1|\xi_4\rangle - \frac{1}{2}|\xi_5\rangle - |\xi_7\rangle - |\lambda_2\rangle + 2|\lambda_3\rangle - \tilde{\gamma}T_0|\lambda_4\rangle, \quad (7.24)$$

$$\delta|\Psi_2\rangle = -L_2|\xi_3\rangle + \frac{1}{2}|\xi_5\rangle + |\xi_7\rangle - 2|\lambda_2\rangle + |\lambda_3\rangle - \tilde{\gamma}T_0|\lambda_4\rangle, \quad (7.25)$$

$$\delta|\Psi_3\rangle = -\tilde{\gamma}T_1|\xi_2\rangle + \frac{1}{2}|\xi_5\rangle - |\xi_6\rangle - |\lambda_3\rangle, \quad (7.26)$$

$$\delta|\Psi_4\rangle = -\tilde{\gamma}T_2|\xi_1\rangle + \frac{1}{2}|\xi_5\rangle - |\xi_6\rangle - 4|\xi_{11}\rangle - |\lambda_2\rangle, \quad (7.27)$$

$$\delta|\Psi_5\rangle = L_1|\xi_2\rangle + |\xi_9\rangle - |\lambda_1\rangle + \tilde{\gamma}T_0|\lambda_3\rangle, \quad (7.28)$$

$$\delta|\Psi_6\rangle = \tilde{\gamma}T_1|\xi_4\rangle + |\xi_8\rangle - |\lambda_4\rangle, \quad (7.29)$$

$$\delta|\Psi_7\rangle = \tilde{\gamma}T_2|\xi_3\rangle + |\xi_8\rangle + 2|\xi_{10}\rangle + |\lambda_4\rangle, \quad (7.30)$$

$$\delta|\Psi_8\rangle = L_2|\xi_1\rangle + |\xi_9\rangle + \tilde{\gamma}T_0|\lambda_2\rangle - |\lambda_1\rangle, \quad (7.31)$$

$$\delta|\Psi_9\rangle = |\xi_5\rangle - 4|\xi_{11}\rangle - T|\xi_{13}\rangle, \quad (7.32)$$

$$\delta|\Psi_{10}\rangle = -\frac{1}{2}|\xi_5\rangle + |\xi_6\rangle + |\xi_7\rangle + L_{11}|\xi_{12}\rangle, \quad (7.33)$$

for the vectors $|\varphi_l\rangle, |\rho_l\rangle$,

$$\delta|\varphi_1\rangle = -2|\xi_2\rangle + \tilde{\gamma}T_1^+|\xi_6\rangle + L_1^+|\xi_8\rangle + \tilde{\gamma}T_1|\xi_{12}\rangle - |\lambda_6\rangle\,, \quad (7.34)$$

$$\delta|\varphi_2\rangle = -L_1^+|\xi_7\rangle - \tilde{\gamma}T_1^+|\xi_9\rangle + L_1|\xi_{12}\rangle - \tilde{\gamma}T_0|\lambda_6\rangle - |\lambda_7\rangle, \quad (7.35)$$

$$\delta|\varphi_3\rangle = -T|\xi_3\rangle + |\xi_4\rangle - \tilde{\gamma}T_1^+|\xi_{10}\rangle\,, \quad (7.36)$$

$$\delta|\varphi_4\rangle = T|\xi_1\rangle - |\xi_2\rangle + L_1^+|\xi_{10}\rangle + 2\tilde{\gamma}T_1^+|\xi_{11}\rangle\,, \quad (7.37)$$

$$\delta|\rho_1\rangle = -\tilde{\gamma}T_1|\xi_1\rangle + T^+|\xi_6\rangle + |\xi_{13}\rangle - |\lambda_9\rangle\,, \quad (7.38)$$

$$\delta|\rho_2\rangle = \tilde{\gamma}T_1|\xi_3\rangle - T^+|\xi_8\rangle\,, \quad (7.39)$$

$$\delta|\rho_3\rangle = L_1|\xi_1\rangle - T^+|\xi_9\rangle - 2|\lambda_8\rangle + \tilde{\gamma}T_0|\lambda_9\rangle\,, \quad (7.40)$$

$$\delta|\rho_4\rangle = L_1|\xi_3\rangle + T^+|\xi_7\rangle - |\xi_{13}\rangle\,, \quad (7.41)$$

and for the vectors $|\chi_m\rangle$,

$$\delta|\chi_1\rangle = -\tilde{\gamma}T_0|\xi_1\rangle - \tilde{\gamma}T_2^+|\lambda_1\rangle + L_2^+|\lambda_2\rangle + T^+|\lambda_7\rangle - 2\tilde{\gamma}T_1^+|\lambda_8\rangle + L_1^+|\lambda_9\rangle\,, \quad (7.42)$$

$$\delta|\chi_2\rangle = -\tilde{\gamma}T_0|\xi_2\rangle - \tilde{\gamma}T_1^+|\lambda_1\rangle + L_1^+|\lambda_3\rangle - |\lambda_7\rangle\,, \quad (7.43)$$

$$\delta|\chi_3\rangle = -2|\xi_1\rangle + \tilde{\gamma}T_0|\xi_3\rangle + \tilde{\gamma}T_2^+|\lambda_3\rangle - L_2^+|\lambda_4\rangle - T^+|\lambda_6\rangle + \tilde{\gamma}T_1^+|\lambda_9\rangle\,, \quad (7.44)$$

$$\delta|\chi_4\rangle = -2|\xi_2\rangle + \tilde{\gamma}T_0|\xi_4\rangle + \tilde{\gamma}T_1^+|\lambda_2\rangle + L_1^+|\lambda_4\rangle + |\lambda_6\rangle\,, \quad (7.45)$$

$$\delta|\chi_5\rangle = \tilde{\gamma}T_0|\xi_5\rangle - 4|\lambda_1\rangle + 2|\lambda_5\rangle\,, \quad (7.46)$$

$$\delta|\chi_6\rangle = \tilde{\gamma}T_0|\xi_6\rangle - 2|\xi_9\rangle - 2|\lambda_1\rangle + |\lambda_5\rangle + \tilde{\gamma}T_1|\lambda_7\rangle\,, \quad (7.47)$$

$$\delta|\chi_7\rangle = \tilde{\gamma}T_0|\xi_7\rangle + 2|\xi_9\rangle - |\lambda_5\rangle + L_1|\lambda_6\rangle\,, \quad (7.48)$$

$$\delta|\chi_8\rangle = -\tilde{\gamma}T_0|\xi_8\rangle - 2|\xi_6\rangle - 2|\xi_7\rangle + 2|\lambda_3\rangle - \tilde{\gamma}T_1|\lambda_6\rangle\,, \quad (7.49)$$

$$\delta|\chi_9\rangle = -\tilde{\gamma}T_0|\xi_9\rangle + L_1|\lambda_7\rangle\,, \quad (7.50)$$

$$\delta|\chi_{10}\rangle = -\tilde{\gamma}T_0|\xi_{10}\rangle - 4|\xi_{11}\rangle + |\lambda_2\rangle + |\lambda_3\rangle - T|\lambda_9\rangle\,, \quad (7.51)$$

$$\delta|\chi_{11}\rangle = \tilde{\gamma}T_0|\xi_{11}\rangle - |\lambda_1\rangle + T|\lambda_8\rangle\,, \tag{7.52}$$
$$\delta|\chi_{12}\rangle = \tilde{\gamma}T_0|\xi_{12}\rangle + L_{11}^+|\lambda_5\rangle + L_1^+|\lambda_6\rangle + \tilde{\gamma}T_1^+|\lambda_7\rangle\,, \tag{7.53}$$
$$\delta|\chi_{13}\rangle = \tilde{\gamma}T_0|\xi_{13}\rangle - T^+|\lambda_5\rangle - 4|\lambda_8\rangle\,. \tag{7.54}$$

Then, substituting expressions (7.15)–(7.18) into (5.30), (5.31), we obtain the gauge transformations for the zero-stage gauge vectors $|\xi_m\rangle, |\lambda_n\rangle$,

$$\delta|\xi_1\rangle = -\tilde{\gamma}T_2^+|\xi_1^{(1)}\rangle + L_2^+|\xi_2^{(1)}\rangle + T^+|\xi_7^{(1)}\rangle - 2\tilde{\gamma}T_1^+|\xi_8^{(1)}\rangle + L_1^+|\xi_9^{(1)}\rangle\,, \tag{7.55}$$
$$\delta|\xi_2\rangle = -\tilde{\gamma}T_1^+|\xi_1^{(1)}\rangle + L_1^+|\xi_3^{(1)}\rangle - |\xi_7^{(1)}\rangle\,, \tag{7.56}$$
$$\delta|\xi_3\rangle = \tilde{\gamma}T_2^+|\xi_3^{(1)}\rangle - L_2^+|\xi_4^{(1)}\rangle - T^+|\xi_6^{(1)}\rangle + \tilde{\gamma}T_1^+|\xi_9^{(1)}\rangle\,, \tag{7.57}$$
$$\delta|\xi_4\rangle = \tilde{\gamma}T_1^+|\xi_2^{(1)}\rangle + L_1^+|\xi_4^{(1)}\rangle + |\xi_6^{(1)}\rangle\,, \tag{7.58}$$
$$\delta|\xi_5\rangle = -4|\xi_1^{(1)}\rangle + 2|\xi_5^{(1)}\rangle\,, \tag{7.59}$$
$$\delta|\xi_6\rangle = -2|\xi_2^{(1)}\rangle + |\xi_5^{(1)}\rangle + \tilde{\gamma}T_1|\xi_7^{(1)}\rangle - |\lambda_2^{(1)}\rangle\,, \tag{7.60}$$
$$\delta|\xi_7\rangle = -|\xi_5^{(1)}\rangle + L_1|\xi_6^{(1)}\rangle + |\lambda_2^{(1)}\rangle\,, \tag{7.61}$$
$$\delta|\xi_8\rangle = 2|\xi_3^{(1)}\rangle - \tilde{\gamma}T_1|\xi_6^{(1)}\rangle\,, \tag{7.62}$$
$$\delta|\xi_9\rangle = L_1|\xi_7^{(1)}\rangle - 2|\lambda_1^{(1)}\rangle - \tilde{\gamma}T_0|\lambda_2^{(1)}\rangle\,, \tag{7.63}$$
$$\delta|\xi_{10}\rangle = |\xi_2^{(1)}\rangle + |\xi_3^{(1)}\rangle - T|\xi_9^{(1)}\rangle\,, \tag{7.64}$$
$$\delta|\xi_{11}\rangle = -|\xi_1^{(1)}\rangle + T|\xi_8^{(1)}\rangle\,, \tag{7.65}$$
$$\delta|\xi_{12}\rangle = L_{11}^+|\xi_5^{(1)}\rangle + L_1^+|\xi_6^{(1)}\rangle + \tilde{\gamma}T_1^+|\xi_7^{(1)}\rangle\,, \tag{7.66}$$
$$\delta|\xi_{13}\rangle = -T^+|\xi_5^{(1)}\rangle - 4|\xi_8^{(1)}\rangle\,, \tag{7.67}$$
$$\delta|\lambda_1\rangle = -\tilde{\gamma}T_0|\xi_1^{(1)}\rangle - 2|\lambda_1^{(1)}\rangle\,, \tag{7.68}$$
$$\delta|\lambda_2\rangle = -2|\xi_1^{(1)}\rangle + \tilde{\gamma}T_0|\xi_2^{(1)}\rangle + |\lambda_2^{(1)}\rangle\,, \tag{7.69}$$
$$\delta|\lambda_3\rangle = -2|\xi_1^{(1)}\rangle + \tilde{\gamma}T_0|\xi_3^{(1)}\rangle + |\lambda_2^{(1)}\rangle\,, \tag{7.70}$$
$$\delta|\lambda_4\rangle = -2|\xi_2^{(1)}\rangle - \tilde{\gamma}T_0|\xi_4^{(1)}\rangle + 2|\xi_3^{(1)}\rangle\,, \tag{7.71}$$
$$\delta|\lambda_5\rangle = -\tilde{\gamma}T_0|\xi_5^{(1)}\rangle - 4|\lambda_1^{(1)}\rangle\,, \tag{7.72}$$
$$\delta|\lambda_6\rangle = -\tilde{\gamma}T_0|\xi_6^{(1)}\rangle - 2|\xi_7^{(1)}\rangle - \tilde{\gamma}T_1^+|\lambda_2^{(1)}\rangle\,, \tag{7.73}$$
$$\delta|\lambda_7\rangle = \tilde{\gamma}T_0|\xi_7^{(1)}\rangle + 2\tilde{\gamma}T_1^+|\lambda_1^{(1)}\rangle + L_1^+|\lambda_2^{(1)}\rangle\,, \tag{7.74}$$
$$\delta|\lambda_8\rangle = -\tilde{\gamma}T_0|\xi_8^{(1)}\rangle + T^+|\lambda_1^{(1)}\rangle\,, \tag{7.75}$$
$$\delta|\lambda_9\rangle = \tilde{\gamma}T_0|\xi_9^{(1)}\rangle - 4|\xi_8^{(1)}\rangle - T^+|\lambda_2^{(1)}\rangle\,. \tag{7.76}$$

Finally, using the equations (5.30), (5.31) for the vectors (7.18)–(7.19), we find the gauge transformations for the first-stage gauge vectors $|\xi_n^{(1)}\rangle, |\lambda_o^{(1)}\rangle$,

$$\delta|\xi_1^{(1)}\rangle = -2|\xi_1^{(2)}\rangle\,, \qquad \delta|\xi_2^{(1)}\rangle = |\xi_2^{(2)}\rangle\,, \tag{7.77}$$
$$\delta|\xi_3^{(1)}\rangle = |\xi_2^{(2)}\rangle\,, \qquad \delta|\xi_4^{(1)}\rangle = 0\,, \tag{7.78}$$
$$\delta|\xi_5^{(1)}\rangle = -4|\xi_1^{(2)}\rangle\,, \qquad \delta|\xi_6^{(1)}\rangle = -\tilde{\gamma}T_1^+|\xi_2^{(2)}\rangle\,, \tag{7.79}$$
$$\delta|\xi_7^{(1)}\rangle = 2\tilde{\gamma}T_1^+|\xi_1^{(2)}\rangle + L_1^+|\xi_2^{(2)}\rangle\,, \qquad \delta|\xi_8^{(1)}\rangle = T^+|\xi_1^{(2)}\rangle\,, \tag{7.80}$$
$$\delta|\xi_9^{(1)}\rangle = -T^+|\xi_1^{(2)}\rangle\,, \tag{7.81}$$

$$\delta|\lambda_1^{(1)}\rangle = \tilde{\gamma}T_0|\xi_1^{(2)}\rangle\,, \qquad \delta|\lambda_2^{(1)}\rangle = -\tilde{\gamma}T_0|\xi_2^{(2)}\rangle - 4|\xi_1^{(2)}\rangle. \tag{7.82}$$

In order to derive the action $\mathcal{S}_{(1,1)}$ (7.22) only in terms of the component fields, we, first of all, present the vectors $|\Psi\rangle$, $|\Psi_k\rangle$, $|\varphi_l\rangle$,$|\rho_l\rangle$, $|\chi_m\rangle$ and the corresponding bra-vectors as expansions with respect to the initial and auxiliary creation operators:

$$\begin{aligned}|\Psi\rangle_{(1,1)} &= \Big(a_1^{+\mu}a_2^{+\nu}\psi_{\mu,\nu}(x) + f_2^+a_1^{+\mu}\tilde{\gamma}\psi_\mu^1(x) + f_1^+a_2^{+\nu}\tilde{\gamma}\psi_\nu^2(x) + b_{12}^+\psi(x) \\ &\quad + f_1^+f_2^+\psi_1^1(x) + \frac{1}{2}a_1^{+\mu}a_1^{+\nu}b^+\psi_{\mu\nu}(x) + a_1^{+\mu}f_1^+b^+\tilde{\gamma}\psi_\mu^3(x) + b_{11}^+b^+\psi_2^2(x)\Big)|0\rangle\,,\end{aligned} \tag{7.83}$$

$$\begin{aligned}{}_{(1,1)}\langle\tilde{\Psi}| &= \langle 0|\Big(\psi_{\mu,\nu}^+(x)a_2^\nu a_1^\mu + \psi_\mu^{1+}(x)\tilde{\gamma}a_1^\mu f_2 + \psi_\nu^{2+}(x)\tilde{\gamma}a_2^\nu f_1 + \psi^+(x)b_{12} \\ &\quad + \psi_1^{1+}(x)f_2f_1 + \frac{1}{2}\psi_{\mu\nu}^+(x)ba_1^\nu a_1^\mu + \psi_\mu^{3+}(x)\tilde{\gamma}bf_1a_1^\mu + \psi_2^{2+}(x)bb_{11}\Big)\tilde{\gamma}_0\,,\end{aligned} \tag{7.84}$$

$$|\chi_m\rangle_{(0,1)} = \Big(a_1^{+\mu}b^+\chi_\mu^{1m}(x) + a_2^{+\mu}\chi_\mu^{2m}(x) + f_1^+b^+\tilde{\gamma}\chi_m^1(x) + f_2^+\tilde{\gamma}\chi_m^2(x)\Big)|0\rangle, m = 1,3, \tag{7.85}$$

$${}_{(0,1)}\langle\tilde{\chi}_m| = \langle 0|\Big(\chi_\mu^{1m+}(x)ba_1^\mu + \chi_\mu^{2m+}(x)a_2^\mu + \chi_m^{1+}(x)\tilde{\gamma}bf_1 + \chi_m^{2+}(x)\tilde{\gamma}f_2\Big)\tilde{\gamma}_0, m = 1,3, \tag{7.86}$$

$$|\chi_{12}\rangle_{(2,0)} = \Big(\frac{1}{2}a_1^{+\mu}a_1^{+\nu}\chi_{\mu\nu}^{12}(x) + a_1^{+\mu}f_1^+\tilde{\gamma}\chi_\mu^{12}(x) + b_{11}^+\chi_{12}(x)\Big)|0\rangle\,, \tag{7.87}$$

$${}_{(2,0)}\langle\tilde{\chi}_{12}| = \langle 0|\Big(\frac{1}{2}\chi_{\mu\nu}^{12+}(x)a_1^\nu a_1^\mu + \chi_\mu^{12+}(x)\tilde{\gamma}f_1a_1^\mu + \chi_{12}^+(x)b_{11}\Big)\tilde{\gamma}_0\,, \tag{7.88}$$

$$|\chi_m\rangle_{(1,0)} = \Big(a_1^{+\mu}\chi_\mu^m(x) + f_1^+\tilde{\gamma}\chi_m(x)\Big)|0\rangle,\ m = 2,4, \tag{7.89}$$

$${}_{(1,0)}\langle\tilde{\chi}_m| = \langle 0|\Big(\chi_\mu^{m+}(x)a_1^\mu + \chi_m^+(x)\tilde{\gamma}f_1\Big)\tilde{\gamma}_0, \tag{7.90}$$

$$|\Psi_k\rangle_{(0,0)} = \psi_k(x)|0\rangle, k = 1,\ldots,10, \qquad |\varphi_l\rangle_{(1,0)} = \Big(a_1^{+\mu}\varphi_\mu^l(x) + f_1^+\tilde{\gamma}\varphi_l(x)\Big)|0\rangle, \tag{7.91}$$

$${}_{(0,0)}\langle\tilde{\Psi}_k| = \langle 0|\psi_k^+(x)\tilde{\gamma}_0, \qquad {}_{(1,0)}\langle\tilde{\varphi}_l| = \langle 0|\Big(\varphi_\mu^{l+}(x)a_1^\mu + \varphi_l^+(x)\tilde{\gamma}f_1\Big)\tilde{\gamma}_0, \tag{7.92}$$

$$|\rho_l\rangle_{(-1,1)} = b^+\rho_l(x)|0\rangle,\ l = 1,..,4, \qquad |\chi_m\rangle_{(0,0)} = \chi_m(x)|0\rangle,\ m = 5,\ldots,11\,, \tag{7.93}$$

$${}_{(-1,1)}\langle\tilde{\rho}_l| = \langle 0|\rho_l^+(x)b\tilde{\gamma}_0, \qquad {}_{(0,0)}\langle\tilde{\chi}_m| = \langle 0|\chi_m^+(x)\tilde{\gamma}_0\,, \tag{7.94}$$

$$|\chi_{13}\rangle_{(-1,1)} = b^+\chi_{13}(x)|0\rangle\,, \qquad {}_{(-1,1)}\langle\tilde{\chi}_{13}| = \langle 0|\chi_{13}^+(x)b\tilde{\gamma}_0\,. \tag{7.95}$$

Second, let us fix preliminarily a part of the gauge ambiguity starting from the first-stage gauge parameters. To this end, we can use the choice of the second-stage independent vectors $|\xi_1^{(2)}\rangle$,$|\xi_2^{(2)}\rangle$, entering relations (7.77)–(7.82) as shift parameters, in order to get rid, for instance, of the vectors $|\xi_1^{(1)}\rangle$, $|\lambda_2^{(1)}\rangle$, so that the description of the model is transformed to a first-stage reducible theory with independent first-stage gauge parameters, $|\lambda_1^{(1)}\rangle$, $|\xi_n^{(1)}\rangle$, $n = 2,\ldots,8$, and without restrictions (7.77)–(7.82).

By the same argument, we can make the zero-stage gauge vectors $|\xi_m\rangle$, $m = 2,4,6,\ldots,10,13$ to vanish by using a choice for the parameters $|\xi_m^{(1)}\rangle$, $m = 7,6,2,5,3$, $|\lambda_1^{(1)}\rangle$, $|\xi_9^{(1)}\rangle$, $|\xi_8^{(1)}\rangle$, respectively, in the gauge transformations (7.55)–(7.76). As a result, the remaining gauge transformations with the independent first-stage gauge parameter $|\xi_4^{(1)}\rangle$ for the remaining zero-stage vectors have the form

$$\delta|\xi_k\rangle = 0,\ \ k = 1,5,11,12\,, \qquad \delta|\xi_3\rangle = -L_2^+|\xi_4^{(1)}\rangle\,, \tag{7.96}$$

$$\delta|\lambda_4\rangle = -\tilde{\gamma}T_0|\xi_4^{(1)}\rangle\,, \qquad \delta|\lambda_l\rangle = 0,\ \ l = 1,2,3,5,\ldots,9\,. \tag{7.97}$$

Finally, in the same manner, we can get rid of the fields $|\Psi_k\rangle$, $k = 1, \ldots, 5$, $|\varphi_1\rangle$, $|\varphi_2\rangle$, $|\rho_1\rangle$, $|\rho_3\rangle$, $|\chi_3\rangle$, $|\chi_5\rangle$ with the help of a corresponding choice for the independent gauge parameters $|\xi_5\rangle$, $|\lambda_2\rangle$, $|\lambda_3\rangle$, $|\xi_{11}\rangle$, $|\lambda_l\rangle$, $l = 1, 6, 7, 9, 8$, $|\xi_1\rangle$, $|\lambda_5\rangle$, respectively, in the gauge transformations (7.23)–(7.54). The resultant gauge transformations for the remaining fields $|\Psi\rangle$, $|\Psi_k\rangle$, $k = 6, \ldots, 10$, $|\varphi_3\rangle$, $|\varphi_4\rangle$, $|\rho_2\rangle$, $|\rho_4\rangle$, $|\chi_k\rangle$, $k = 1, 2, 4, 6, \ldots, 13$, with the zero-stage gauge vectors $|\xi_3\rangle$, $|\xi_{12}\rangle$, $|\lambda_4\rangle$, that have not been used previously, are reduced to

$$\delta|\Psi\rangle = L_1^+|\xi_3\rangle + T^+|\xi_{12}\rangle\,, \tag{7.98}$$

$$\delta|\Psi_6\rangle = -|\lambda_4\rangle\,, \qquad \delta|\Psi_7\rangle = \tilde{\gamma}T_2|\xi_3\rangle + |\lambda_4\rangle\,, \tag{7.99}$$

$$\delta|\Psi_l\rangle = 0,\ l = 8, 9\,, \qquad \delta|\Psi_{10}\rangle = L_{11}|\xi_{12}\rangle\,, \tag{7.100}$$

$$\delta|\varphi_3\rangle = -T|\xi_3\rangle\,, \qquad \delta|\varphi_4\rangle = 0\,, \tag{7.101}$$

$$\delta|\rho_2\rangle = \tilde{\gamma}T_1|\xi_3\rangle\,, \qquad \delta|\rho_4\rangle = L_1|\xi_3\rangle\,, \tag{7.102}$$

$$\delta|\chi_k\rangle = 0,\ k = 1, 2, 6, \ldots, 11, 13\,, \qquad \delta|\chi_4\rangle = L_1^+|\lambda_4\rangle\,, \tag{7.103}$$

$$\delta|\chi_{12}\rangle = \tilde{\gamma}T_0|\xi_{12}\rangle\,. \tag{7.104}$$

As a consequence, the action (7.22) is simplified as follows:

$$\begin{aligned}
\mathcal{S}_{(1,1)} = \Big[&\langle\tilde{\Psi}|K_{(1,1)}\left\{\tfrac{1}{2}T_0|\Psi\rangle + T_1^+|\chi_1\rangle + T_2^+|\chi_2\rangle + \tilde{\gamma}L_2^+|\chi_4\rangle + \tilde{\gamma}T^+|\chi_{12}\rangle + \tilde{\gamma}L_{11}^+|\chi_{13}\rangle\right\}\\
&+\langle\tilde{\Psi}_6|K_{(1,1)}\left\{-T_0|\Psi_8\rangle + \tilde{\gamma}L_2|\chi_1\rangle + \tilde{\gamma}|\chi_9\rangle\right\}\\
&+\langle\tilde{\Psi}_7|K_{(1,1)}\left\{\tilde{\gamma}L_1|\chi_2\rangle + \tilde{\gamma}|\chi_9\rangle\right\} + \langle\tilde{\Psi}_8|K_{(1,1)}\left\{-T_1|\chi_4\rangle + \tilde{\gamma}|\chi_8\rangle\right\}\\
&+\langle\tilde{\Psi}_9|K_{(1,1)}\left\{T_0|\Psi_{10}\rangle + \tilde{\gamma}|\chi_6\rangle + \tilde{\gamma}|\chi_7\rangle + \tilde{\gamma}L_{11}|\chi_{12}\rangle\right\}\\
&-\langle\tilde{\Psi}_{10}|K_{(1,1)}\left\{4\tilde{\gamma}|\chi_{11}\rangle + \tilde{\gamma}T|\chi_{13}\rangle\right\}\\
&+\langle\tilde{\varphi}_3|K_{(1,1)}\left\{-\tilde{\gamma}L_1^+|\chi_7\rangle + T_1^+|\chi_9\rangle + \tilde{\gamma}L_1|\chi_{12}\rangle\right\}\\
&+\langle\tilde{\varphi}_4|K_{(1,1)}\left\{-2\tilde{\gamma}|\chi_2\rangle - T_1^+|\chi_6\rangle - T_1|\chi_{12}\rangle\right\}\\
&+\langle\tilde{\rho}_2|K_{(1,1)}\left\{\tilde{\gamma}L_1|\chi_1\rangle - \tilde{\gamma}T^+|\chi_9\rangle\right\} + \langle\tilde{\rho}_4|K_{(1,1)}\left\{-\tfrac{1}{2}T_0|\rho_4\rangle - \tilde{\gamma}T^+|\chi_7\rangle + \tilde{\gamma}|\chi_{13}\rangle\right\}\\
&+\langle\tilde{\chi}_2|K_{(1,1)}\left\{-\tilde{\gamma}|\chi_4\rangle\right\} + \langle\tilde{\chi}_4|K_{(1,1)}\left\{-\tfrac{1}{2}T_0|\chi_4\rangle\right\} + \langle\tilde{\chi}_6|K_{(1,1)}\left\{-\tilde{\gamma}|\chi_{10}\rangle\right\}\\
&+\langle\tilde{\chi}_7|K_{(1,1)}\left\{-\tilde{\gamma}|\chi_{10}\rangle\right\} + \langle\tilde{\chi}_8|K_{(1,1)}\left\{T_0|\chi_{10}\rangle - 2\tilde{\gamma}|\chi_{11}\rangle\right\}\Big] + c.c.\,,
\end{aligned} \tag{7.105}$$

Then, after a substitution of expressions (7.83)–(7.95) into (7.105), we find the action

for a field of spin $(3/2, 3/2)$ in a manifest form:

$$
\begin{aligned}
\mathcal{S}_{(1,1)} = \int d^dx \Bigg[& \bar\psi_{\mu,\nu}\Bigg\{-\frac{\imath}{2}\gamma^\rho\partial_\rho\psi^{\mu,\nu} + \gamma^\mu\chi^{21\nu} + \gamma^\nu\chi^{2\mu} - \imath\partial^\nu\chi^{4\mu} - \chi^{12\mu\nu}\Bigg\} \\
& -2\bar\psi_{\mu\nu}\Bigg\{-\frac{\imath}{2}\gamma^\rho\partial_\rho\psi^{\mu\nu} + \gamma^\mu\chi^{11\nu} + \frac{1}{2}\chi^{12\mu\nu} + \frac{1}{2}\eta^{\mu\nu}\chi_{13}\Bigg\} + \bar\psi^1_\mu\Bigg\{\imath\left(3-\frac{d}{2}\right)\gamma^\rho\partial_\rho\psi^{1\mu} \\
& +4\imath\gamma^\rho\partial_\rho\psi^{3\mu} - \gamma^\mu(4\chi^1_1 + (6-d)\chi^2_1) + (6-d)\chi^{2\mu} + 4\chi^{11\mu} + (10-d)\chi^{12\mu}\Bigg\} \\
& +\bar\psi^3_\mu\big\{\imath d\gamma^\rho\partial_\rho\psi^{3\mu} - \gamma^\mu(2d\chi^1_1 + 4\chi^2_1) + 4\chi^{2\mu} + 2d\chi^{11\mu} + (4+2d)\chi^{12\mu}\big\} \\
& +(d-2)\bar\psi^2_\mu\Bigg\{-\frac{\imath}{2}\gamma^\rho\partial_\rho\psi^{2\mu} + \gamma^\mu\chi_2 - \chi^{21\mu} + \imath\partial^\mu\chi_4 + \chi^{12\mu}\Bigg\} \\
& +\bar\psi\Bigg\{\frac{\imath}{2}\left(\frac{d}{4}-1\right)\gamma^\rho\partial_\rho\psi + \imath\left(\frac{d}{2}-3\right)\gamma^\rho\partial_\rho\psi^1_1 - \imath\gamma^\rho\partial_\rho\psi^2_2 + 2\chi^1_1 + \left(1-\frac{d}{2}\right)\chi_2 \\
& +\left(3-\frac{d}{2}\right)\chi^2_1 + \left(\frac{d}{2}-1\right)\chi_{12} + \chi_{13}\Bigg\} + \bar\psi^1_1\big\{-\frac{\imath}{2}\gamma^\rho\partial_\rho\big[(20-8d+d^2)\psi^1_1 + 8\psi^2_2\big] \\
& +8\chi^1_1 + (20-8d+d^2)\chi^2_1 - (8-6d+d^2)\chi_2 + (d-2)\chi_{12} + 4\chi_{13}\big\} \\
& +\bar\psi^2_2\Bigg\{-\frac{\imath d}{2}\gamma^\rho\partial_\rho\psi^2_2 + 2d\chi^1_1 + 4\chi^2_1 + d(\chi_{12}+\chi_{13})\Bigg\} \\
& +\bar\psi_6\big\{\imath\gamma^\mu\partial_\mu\psi_8 + \imath\partial^\mu\chi^{21}_\mu + \chi_9\big\} + \bar\psi_7\big\{\imath\partial^\mu\chi^2_\mu + \chi_9\big\} + \bar\psi_8\big\{\gamma^\mu\chi^4_\mu + (2-d)\chi_4 + \chi_8\big\} \\
& +\bar\psi_9\Bigg\{-\imath\gamma^\mu\partial_\mu\psi_{10} + \chi_6 + \chi_7 + \frac{1}{2}\chi^{12\mu}_\mu + \left(1-\frac{d}{2}\right)\chi_{12}\Bigg\} + \bar\psi_{10}\Bigg\{-4\chi_{11} + 2\chi_{13}\Bigg\} \\
& +\bar\varphi^3_\mu\big\{-\imath\partial^\mu\chi_7 - \gamma^\mu\chi_9 - \imath\partial_\nu\chi^{12\mu\nu}\big\} + (d-2)\bar\varphi_4\big\{-2\chi_2 - \chi_6 - \gamma^\mu\chi^{12}_\mu + \chi_{12}\big\} \\
& +\bar\varphi^4_\mu\big\{2\chi^{2\mu} + \gamma^\mu\chi_6 - \gamma_\nu\chi^{12\mu\nu} + (d-2)\chi^{12\mu}\big\} - 2\bar\rho_2\big\{\imath\partial^\mu\chi^{11}_\mu - \chi_9\big\} \\
& -2\bar\rho_4\Bigg\{-\frac{\imath}{2}\gamma^\mu\partial_\mu\rho_4 - \chi_7 + \chi_{13}\Bigg\} + \bar\chi^2_\mu\chi^{4\mu} + (2-d)\bar\chi^4_2\chi_4 - \frac{\imath}{2}\bar\chi^4_\mu\gamma^\nu\partial_\nu\chi^{4\mu} \\
& +i\left(1-\frac{d}{2}\right)\bar\chi_4\gamma^\nu\partial_\nu\chi_4 - \bar\chi_6\chi_{10} - \bar\chi_7\chi_{10} - \bar\chi_8\big\{2\chi_{11} + \imath\gamma^\mu\partial_\mu\chi_{10}\big\}\Bigg] + c.c. \qquad (7.106)
\end{aligned}
$$

Here, the d^2 functions $\psi_{\mu,\nu}$ are the basic spin-tensor field of spin $(3/2, 3/2)$ subject to the conditions (2.2)–(2.5) encoded by the action, and the other fields are auxiliary ones. Then, in order to present the gauge transformations in a manifest form, we rewrite the zero-stage $|\xi_3\rangle, |\lambda_4\rangle$ and first-stage $|\xi^{(1)}_4\rangle$ gauge parameters in (7.15)–(7.17) as follows:

$$
\begin{aligned}
|\xi_3\rangle_{(0,1)} &= \Big(a^{+\mu}_1 b^+\xi^{13}_\mu(x) + a^{+\mu}_2\xi^{23}_\mu(x) + f^+_1 b^+\tilde\gamma\xi^1_3(x) + f^+_2\tilde\gamma\xi^2_3(x)\Big)|0\rangle, \qquad (7.107)\\
|\xi_{12}\rangle_{(2,0)} &= \Big(\tfrac{1}{2}a^{+\mu}_1 a^{+\nu}_1\xi^{12}_{\mu\nu}(x) + a^{+\mu}_1 f^+_1\tilde\gamma\xi^{12}_\mu(x) + b^+_{11}\xi_{12}(x)\Big)|0\rangle\,, \qquad (7.108)\\
|\lambda_4\rangle_{(0,0)} &= \lambda_4(x)|0\rangle\,, \qquad |\xi^{(1)}_4\rangle_{(0,0)} = \xi^{(1)}_4(x)|0\rangle\,, \qquad (7.109)
\end{aligned}
$$

and then substitute them into (7.98)–(7.104). As a result, the gauge transformations for the fields associated with spin $(3/2, 3/2)$ have the form

$$\delta\psi_{\mu,\nu}(x) = -\imath\partial_\mu\xi^{23}_\nu(x) - \xi^{12}_{\mu\nu}(x)\,, \qquad \delta\psi_{\mu\nu}(x) = -\imath\partial_{\{\mu}\xi^{13}_{\nu\}}(x) - \xi^{12}_{\mu\nu}(x)\,, \tag{7.110}$$

$$\delta\psi^1_\mu(x) = -\imath\partial_\mu\xi^2_3(x) - \xi^{12}_\mu(x)\,, \qquad \delta\psi^2_\mu(x) = \ -\xi^{12}_\mu(x)\,, \tag{7.111}$$

$$\delta\psi^3_\mu(x) = -\imath\partial_\mu\xi^1_3(x) + \xi^{12}_\mu(x)\,, \qquad \delta\psi(x) = -2\xi_{12}(x)\,, \tag{7.112}$$

$$\delta\psi^1_1(x) = 0\,, \qquad \delta\psi^2_2(x) = \xi_{12}(x)\,, \tag{7.113}$$

$$\delta\psi_6(x) = -\lambda_4(x)\,, \qquad \delta\psi_{10}(x) = \tfrac{1}{2}\xi^{12\mu}_\mu(x) + (1-\tfrac{d}{2})\xi_{12}(x)\,, \tag{7.114}$$

$$\delta\varphi^3_\mu(x) = \xi^{23}_\mu(x) + 2\xi^{13}_\mu(x)\,, \qquad \delta\varphi_3(x) = 2\xi^1_3(x) + \xi^2_3(x)\,, \tag{7.115}$$

$$\delta\rho_2(x) = \gamma^\mu\xi^{13}_\mu - d\xi^1_3(x) - 2\xi^2_3(x)\,, \qquad \delta\rho_4(x) = \imath\partial^\mu\xi^{13}_\mu(x)\,, \tag{7.116}$$

$$\delta\chi^4_\mu(x) = -\imath\partial_\mu\lambda_4(x)\,, \qquad \delta\chi_4(x) = 0\,, \tag{7.117}$$

$$\delta\chi^{12}_{\mu\nu}(x) = \imath\gamma^\rho\partial_\rho\xi^{12}_{\mu\nu}(x)\,, \qquad \delta\chi^{12}_\mu(x) = \imath\gamma^\rho\partial_\rho\xi^{12}_\mu(x)\,, \tag{7.118}$$

$$\delta\chi_{12}(x) = \imath\gamma^\rho\partial_\rho\xi_{12}(x)\,, \tag{7.119}$$

$$\delta\psi_7(x) = \gamma^\mu\xi^{23}_\mu(x) + (6-d)\xi^2_3(x) + 4\xi^1_3(x) + \lambda_4(x)\,, \tag{7.120}$$

whereas the nontrivial gauge transformations for the gauge parameters are written as follows:

$$\delta\xi^{23}_\mu(x) = \imath\partial_\mu\xi^{(1)}_4(x)\,, \qquad \delta\lambda_4(x) = -\imath\gamma^\mu\partial_\mu\xi^{(1)}_4(x)\,. \tag{7.121}$$

Let us present the action in terms of the spin-tensor field $\psi_{\mu,\nu}$ alone. To this end, we remove the fields $\psi_{\mu\nu}$, ψ^1_μ, ψ, φ^3_μ, φ_3, ρ_2 by using their gauge transformations with the respective gauge parameters $\xi^{12}_{\mu\nu}$, ξ^{12}_μ, ξ_{12}, ξ^{13}_μ, ξ^1_3, ξ^2_3. Third, we express the fields ψ^1_1, ψ^2_2, χ_8, χ_9 by using the equations of motion $(\psi^1_1, \psi^2_2, \chi_8, \chi_9) = (0, \gamma^\mu\psi^3_\mu, -2\psi_{10}, -\imath\partial^\mu\chi^2_\mu)$, which results in the vanishing of the terms with the Lagrangian multipliers $\chi^1_1, \chi^2_1, \chi_{11}, \psi_7$. As a result, we find

$$\begin{aligned}
\mathcal{S}_{(1,1)} = \int d^dx\Big[&\bar\psi_{\mu,\nu}\Big\{-\frac{\imath}{2}\gamma^\rho\partial_\rho\psi^{\mu,\nu} + \gamma^\mu\chi^{21\nu} + \gamma^\nu\chi^{2\mu} - \imath\partial^\nu\chi^{4\mu} - \chi^{12\mu\nu}\Big\}\\
&+\bar\psi^3_\mu\Big\{\imath d\gamma^\rho\partial_\rho\psi^{3\mu} + \gamma^\mu\Big[-\frac{\imath d}{2}\gamma^\rho\partial_\rho\gamma^\nu\psi^3_\nu + d(\chi_{12}+\chi_{13})\Big] + 4\chi^{2\mu} + 2d\chi^{11\mu}\\
&+(4+2d)\chi^{12\mu}\Big\} + (d-2)\bar\psi^2_\mu\Big\{-\frac{\imath}{2}\gamma^\rho\partial_\rho\psi^{2\mu} + \gamma^\mu\chi_2 - \chi^{21\mu} + \imath\partial^\mu\chi_4 + \chi^{12\mu}\Big\}\\
&+\bar\psi_6\Big\{\imath\gamma^\mu\partial_\mu\psi_8 + \imath\partial^\mu\chi^{21}_\mu - \imath\partial^\mu\chi^2_\mu\Big\} + \bar\psi_8\Big\{\gamma^\mu\chi^4_\mu + (2-d)\chi_4 - 2\psi_{10}\Big\}\\
&+\bar\psi_9\Big\{-\imath\gamma^\mu\partial_\mu\psi_{10} + \chi_6 + \chi_7 + \frac{1}{2}\chi^{12\mu}_\mu + \Big(1-\frac{d}{2}\Big)\chi_{12}\Big\}\\
&+(d-2)\bar\varphi_4\Big\{-2\chi_2 - \chi_6 - \gamma^\mu\chi^{12}_\mu + \chi_{12}\Big\} + \bar\varphi^4_\mu\Big\{2\chi^{2\mu} + \gamma^\mu\chi_6 - \gamma_\nu\chi^{12\mu\nu} + (d-2)\chi^{12\mu}\Big\}\\
&-2\bar\rho_4\Big\{-\frac{\imath}{2}\gamma^\mu\partial_\mu\rho_4 - \chi_7 + \chi_{13}\Big\} + (2-d)\bar\chi_2\chi_4 - \frac{\imath}{2}\bar\chi^4_\mu\gamma^\nu\partial_\nu\chi^{4\mu} + \bar\chi^2_\mu\chi^{4\mu}\\
&+\imath\Big(1-\frac{d}{2}\Big)\bar\chi_4\gamma^\nu\partial_\nu\chi_4 - \bar\chi_6\chi_{10} - \bar\chi_7\chi_{10} + 2\bar\psi_{10}\Big\{\chi_{13} + \imath\gamma^\mu\partial_\mu\chi_{10}\Big\}\Big] + c.c.,
\end{aligned} \tag{7.122}$$

and the remaining reducible nontrivial gauge transformations for the fields associated with spin $(3/2, 3/2)$ have the form

$$\delta\psi_{\mu,\nu} \;=\; -\imath\partial_\mu\xi^{23}_\nu\,, \qquad \delta\psi_6 \;=\; -\lambda_4\,, \qquad \delta\chi^4_\mu \;=\; -\imath\partial_\mu\lambda_4\,. \tag{7.123}$$

We now express the fields ψ^μ_3, φ_4, ψ_{10} by using the corresponding equations of motion $(\psi^\mu_3, \varphi_4, \psi_{10}) = (0, \frac{1}{2}\psi_9, \rho_4)$, which implies the vanishing of the terms with the Lagrangian multipliers $\chi^{11}_\mu, \chi^{12}, \chi_{13}$. We then proceed to do the same with the fields ψ^2_ν, ψ_9, χ_{10}, ρ_4 by using the corresponding equations of motion

$$\psi^2_\nu = \frac{1}{d-2}(\gamma^\mu\psi_{\mu,\nu} + \imath\partial_\nu\psi_6)\,, \qquad \psi_9 = \frac{2}{d}(\eta^{\mu\nu}\psi_{\mu,\nu} + \gamma^\nu\varphi^4_\nu)\,, \tag{7.124}$$

$$\chi_{10} = \frac{4}{d}\left[\left(1-\frac{d}{4}\right)\eta^{\mu\nu}\psi_{\mu,\nu} + \gamma^\mu\varphi^4_\mu\right], \qquad \rho_4 = \frac{1}{d}\left[\left(1-\frac{d}{2}\right)\eta^{\mu\nu}\psi_{\mu,\nu} + \gamma^\nu\varphi^4_\nu\right], \tag{7.125}$$

which results in the fact that the action no longer has any terms containing the Lagrangian multipliers $\chi^{21}_\mu, \chi^{12}_{\mu\nu}, \chi_6, \chi_7$.

Then, the fields $\imath\partial_\mu\psi_6$, φ^4_ν, χ_4, χ^4_μ can be expressed in terms of $\psi_{\mu,\nu}$ by using the corresponding equations of motion for the Lagrangian multipliers χ^2_μ, χ_2, χ^{12}_μ, ψ_8, as follows:

$$\imath\partial_\mu\psi_6 = -\gamma^\rho\psi_{\rho,\mu} + \frac{\gamma_\mu(d-2)}{d^2(d^2-3d+8)}\Big[(d^2+11d-6)\psi_{\rho,\sigma}\eta^{\rho\sigma} - d(d-1)\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}\Big]\,, \tag{7.126}$$

$$\begin{aligned}\chi^4_\mu = &-\frac{\gamma_\mu}{d^2(d^2-3d+8)}\Big[(d^3-15d^2+16d-12)\psi_{\rho,\sigma}\eta^{\rho\sigma} + d(d^2-d+2)\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}\Big] \\ &-\gamma^\rho(\psi_{\rho,\mu} + \psi_{\mu,\rho})\,,\end{aligned} \tag{7.127}$$

$$\varphi^4_\mu = \frac{\gamma_\mu}{d(d^2-3d+8)}\Big[(d^2-3d-6)\psi_{\rho,\sigma}\eta^{\rho\sigma} + d\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}\Big]\,, \tag{7.128}$$

$$\chi_4 = -\frac{1}{d(d^2-3d+8)}\Big[d(d+1)\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma} + (3d^2-23d+10)\psi_{\rho,\sigma}\eta^{\rho\sigma}\Big]\,. \tag{7.129}$$

This causes the terms with the Lagrangian multipliers χ^2_μ, χ_2, χ^{12}_μ, ψ_8 to vanish, so that the action in terms of the rank-2 spin-tensor field $\psi_{\mu,\nu}$ alone takes the form

$$\begin{aligned}\mathcal{S}_{(1,1)} = \int d^dx\Bigg[\bar\psi_{\mu,\nu}\Big\{&-\frac{\imath}{2}\gamma^\rho\partial_\rho\psi^{\mu,\nu} + \imath\partial^\nu\Big[\frac{\gamma^\mu(d^2-d+2)}{d^2(d^2-3d+8)}\Big[(d^2-3d-6)\psi_{\rho,\sigma}\eta^{\rho\sigma} + d\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}\Big] \\ &+\gamma_\rho(\psi^{\rho,\mu} + \psi^{\mu,\rho}) - \frac{1}{d}(d-2)\gamma^\mu\psi_{\rho,\sigma}\eta^{\rho\sigma}\Big]\Big\} \\ &+\frac{d-2}{d}\Big(\eta^{\rho\sigma}\bar\psi_{\rho,\sigma}\gamma_\nu + \frac{(1-d)}{d(d^2-3d+8)}\Big[(d^2-3d-6)\bar\psi_{\rho,\sigma}\eta^{\rho\sigma} + d\bar\psi_{\rho,\sigma}\gamma^\sigma\gamma^\rho\Big]\gamma_\nu\Big)\times\end{aligned}$$

$$\times\Big\{-\frac{\imath}{2}\gamma^\rho\partial_\rho\Big(\frac{d-2}{d}\gamma^\nu\psi_{\rho,\sigma}\eta^{\rho\sigma}+\frac{\gamma^\nu(2-d)(d-1)}{d^2(d^2-3d+8)}\Big[(d^2-3d-6)\psi_{\rho,\sigma}\eta^{\rho\sigma}+d\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}\Big]\Big)$$
$$-\imath\partial^\nu\frac{1}{d(d^2-3d+8)}\Big[d(d+1)\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}+(3d^2-23d+10)\psi_{\rho,\sigma}\eta^{\rho\sigma}\Big]\Big\}$$
$$+\frac{2}{d}\Big(\bar\psi_{\rho,\nu}\eta^{\rho\nu}+\frac{1}{(d^2-3d+8)}\Big[(d^2-3d-6)\bar\psi_{\rho,\sigma}\eta^{\rho\sigma}+d\bar\psi_{\rho,\sigma}\gamma^\sigma\gamma^\rho\Big]\Big)\Big\{-\imath\gamma^\mu\partial_\mu\times$$
$$\times\frac{1}{d}\Big[\Big(1-\frac{d}{2}\Big)\eta^{\rho\sigma}\psi_{\rho,\sigma}+\frac{1}{(d^2-3d+8)}\Big[(d^2-3d-6)\psi_{\rho,\sigma}\eta^{\rho\sigma}+d\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}\Big]\Big]\Big\}$$
$$+\frac{2}{d}\Big\{\Big(1-\frac{d}{2}\Big)\eta^{\mu\nu}\bar\psi_{\mu,\nu}+\frac{1}{(d^2-3d+8)}\Big[(d^2-3d-6)\bar\psi_{\rho,\sigma}\eta^{\rho\sigma}+d\bar\psi_{\rho,\sigma}\gamma^\sigma\gamma^\rho\Big]\Big\}\times$$
$$\times\Big\{\frac{\imath}{2}\gamma^\rho\partial_\rho\frac{1}{d}\Big[\Big(1-\frac{d}{2}\Big)\eta^{\mu\nu}\psi_{\mu,\nu}+\frac{1}{(d^2-3d+8)}\Big[(d^2-3d-6)\psi_{\rho,\sigma}\eta^{\rho\sigma}+d\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}\Big]\Big]$$
$$+\imath\gamma^\rho\partial_\rho\frac{4}{d}\Big[\Big(1-\frac{d}{4}\Big)\eta^{\mu\nu}\psi_{\mu,\nu}+\frac{1}{(d^2-3d+8)}\Big[(d^2-3d-6)\psi_{\rho,\sigma}\eta^{\rho\sigma}+d\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}\Big]\Big]\Big\}$$
$$+\Big\{\frac{(d^2-d+2)}{d^2(d^2-3d+8)}\Big[(d^2-3d-6)\bar\psi_{\rho,\sigma}\eta^{\rho\sigma}+d\bar\psi_{\rho,\sigma}\gamma^\sigma\gamma^\rho\Big]\gamma_\mu+(\bar\psi_{\rho,\mu}+\bar\psi_{\mu,\rho})\gamma^\rho$$
$$-\frac{d-2}{d}\bar\psi_{\rho,\sigma}\eta^{\rho\sigma}\gamma_\mu\Big\}\Big(-\frac{\imath}{2}\gamma^\nu\partial_\nu\Big)\Big\{\frac{\gamma^\mu(d^2-d+2)}{d^2(d^2-3d+8)}\Big[(d^2-3d-6)\psi_{\rho,\sigma}\eta^{\rho\sigma}+d\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}\Big]$$
$$+\gamma_\rho(\psi^{\rho,\mu}+\psi^{\mu,\rho})-\frac{d-2}{d}\gamma^\mu\psi_{\rho,\sigma}\eta^{\rho\sigma}\Big\}$$
$$+\Big(1-\frac{d}{2}\Big)\frac{1}{d^2(d^2-3d+8)^2}\Big[d(d+1)\bar\psi_{\rho,\sigma}\gamma^\sigma\gamma^\rho+(3d^2-23d+10)\bar\psi_{\rho,\sigma}\eta^{\rho\sigma}\Big](\imath\gamma^\nu\partial_\nu)\times$$
$$\times\Big[d(d+1)\gamma^\rho\gamma^\sigma\psi_{\rho,\sigma}+(3d^2-23d+10)\psi_{\rho,\sigma}\eta^{\rho\sigma}\Big]\Big]+c.c.\,, \tag{7.130}$$

which is invariant with respect to the remaining gauge transformations from (7.123), $\delta\psi_{\mu,\nu}(x)=-\imath\partial_\mu\xi^{23}_\nu(x)$, being reducible due to (7.121), $\delta\xi^{23}_\mu(x)\;=\;\imath\partial_\mu\xi^{(1)}_4(x)$.

Finally, extracting the physical, antisymmetric, part $\psi_{[\mu\nu]}$, of the basic field $\psi_{\mu,\nu}$ (thus resolving the condition (2.5) in question), which is in one-to-one correspondence with the Young tableaux (2.1) for $n_1=n_2=1$, we obtain the action and reducible gauge transformation for the spin-tensor $\psi_{[\mu\nu]}$, respectively [for $\gamma^{\rho\sigma}=\frac{1}{2}(\gamma^\rho\gamma^\sigma-\gamma^\sigma\gamma^\rho)$],

$$\mathcal{S}_{\psi_{[\mu\nu]}}=\int d^dx\Big[-\imath\bar\psi_{[\mu\nu]}\gamma^\rho\partial_\rho\psi^{[\mu\nu]}+\imath\frac{(d^2-d+2)}{d(d^2-3d+8)}\bar\psi_{[\mu\nu]}\Big\{\partial^\nu\gamma^\mu\gamma^{\rho\sigma}+\gamma^{\nu\mu}\partial^\sigma\gamma^\rho\Big\}\psi_{[\rho\sigma]}$$
$$+\imath\Big\{\frac{(d-2)\Big[(d-2)^2(d-1)^2+(d^2-d+2)^2\Big]+d^2\Big[14+(2-d)(d+1)^2\Big]}{d^2(d^2-3d+8)^2}$$
$$-\frac{2(d+1)(d-2)^2(d-1)^2}{d^2(d^2-3d+8)^3}\Big\}\bar\psi_{[\rho\sigma]}\gamma^{\sigma\rho}\gamma^\nu\partial_\nu\gamma^{\tau\omega}\psi_{[\tau\omega]}\Big], \tag{7.131}$$

$$\delta\psi_{[\mu\nu]}=-\imath\partial_\mu\xi_\nu+\imath\partial_\nu\xi_\mu,\qquad\delta\xi_\mu=\imath\partial_\mu\xi^{(1)},\qquad(\xi_\mu,\xi^{(1)})\equiv(\xi^{23}_\mu,\xi^{(1)}_4). \tag{7.132}$$

To obtain a Lagrangian description of the massive rank-2 antisymmetric spin-tensor $\psi_{[\mu\nu]}$, having the Young tableaux (2.1) and subject to conditions (2.3), (2.4) and the requirement $(\imath\gamma^\mu\partial_\mu-m)\Phi_{\mu,\nu}(x)=0$, instead of (2.2), we may follow the example of section 7.1

and apply the prescription (5.34)–(5.36) starting directly from the action (7.131) written for a $(d+1)$-dimensional Minkowski space. First, we have the following representation for the fields, $\psi^{[MN]} = (\psi^{[\mu\nu]}, \psi^{[\mu d]})$, with the Stueckelberg field being $\psi^{[\mu d]} \equiv \varphi^{\mu}$, and with the gauge parameters $\xi_N = (\xi_\nu, \xi)$. Second, the corresponding action can be obtained from (7.131) by dimensional projection $\mathbb{R}^{1,d} \to \mathbb{R}^{1,d-1}$ and must be invariant with respect to the gauge transformations

$$\delta\psi_{[\mu\nu]}(x) \;=\; -\imath\partial_{[\mu}\xi_{\nu]}(x)\,, \qquad\qquad \delta\varphi_{\mu}(x) \;=\; -\imath\partial_{\mu}\xi(x) + m\xi_{\mu}(x)\,, \tag{7.133}$$

which, in turn, are reducible:

$$\delta\xi_{\mu}(x) \;=\; \imath\partial_{\mu}\xi^{(1)}(x)\,, \qquad\qquad \delta\xi(x) \;=\; m\xi^{(1)}(x)\,. \tag{7.134}$$

Third, due to (5.36), we use the identity $\gamma^{RS}\psi_{[RS]} = \gamma^{\rho\sigma}\psi_{[\rho\sigma]}$, and the identification

$$i\gamma^{M}\partial_{M}\psi^{[NK]} = (i\gamma^{\mu}\partial_{\mu} - m)\psi^{[NK]}, \qquad i\gamma^{M}\partial_{M}\gamma_{N}\psi^{[NK]} = (i\gamma^{\mu}\partial_{\mu} + m)\gamma_{N}\psi^{[NK]}, \tag{7.135}$$

which holds true if we replace $\psi^{[NK]}$ $[\gamma_N\psi^{[NK]}]$ by the quantities $(\gamma_L)^{2k}\psi^{[NK]}$ $[(\gamma_L)^{2k+1}\psi^{[NK]}]$, for $k \in \mathbb{N}_0$. After removing the gauge parameter $\xi(x)$ by the shift transformation, and then, in the same manner, the field φ_μ, by means of the now independent gauge transformation with the parameter $\xi_\mu(x)$, we obtain the Lagrangian for a massive antisymmetric spin-tensor $(3/2, 3/2)$ field in a d-dimensional flat space:

$$\begin{aligned}\mathcal{L}^{m}_{\psi_{[\mu\nu]}} \;=\; & -\bar{\psi}_{[\mu\nu]}(\imath\gamma^{\rho}\partial_{\rho} - m)\psi^{[\mu\nu]} + \imath\frac{(d^2+d+2)}{(d+1)(d^2-d+6)}\bar{\psi}_{[\mu\nu]}\Big\{\partial^{\nu}\gamma^{\mu}\gamma^{\rho\sigma} + \gamma^{\nu\mu}\partial^{\sigma}\gamma^{\rho}\Big\}\psi_{[\rho\sigma]} \\ & + \Bigg\{\frac{(d-1)\Big[(d-1)^2d^2 + (d^2+d+2)^2\Big] + (d+1)^2\Big[14 + (1-d)(d+2)^2\Big]}{(d+1)^2(d^2-d+6)^2} \\ & - \frac{2(d+2)(d-1)^2d^2}{(d+1)^2(d^2-d+6)^3}\Bigg\}\bar{\psi}_{[\rho\sigma]}\gamma^{\sigma\rho}(\imath\gamma^{\nu}\partial_{\nu} - m)\gamma^{\tau\omega}\psi_{[\tau\omega]}\,.\end{aligned} \tag{7.136}$$

8. Conclusion

In the present work, we have constructed a gauge-invariant Lagrangian description of free half-integer HS fields belonging to an irreducible representation of the Poincare group $ISO(1, d-1)$ with the corresponding Young tableaux having two rows in the "metric-like" formulation. The results of this study are the most general ones and apply to both massive and massless fermionic HS fields with a mixed symmetry in a Minkowski space of any dimension.

In the standard manner, starting from an embedding of fermionic HS fields into vectors of an auxiliary Fock space, we treat the fields as coordinates of Fock-space vectors and reformulate the theory in such terms. We realize the conditions that determine an irreducible Poincare-group representation with a given mass and generalized spin in terms of differential operator constraints imposed on the Fock space vectors. These constraints generate a closed Lie superalgebra of HS symmetry, which contains, with the exception of two basis generators of its Cartan subalgebra, a system of first- and second-class constraints.

JHEP10(2007)040

We demonstrate that the construction of a correct Lagrangian description requires a deformation of the initial symmetry superalgebra, in order to obtain from the system of mixed-class constraints a converted system with the same number of first-class constraints alone, whose structure provides the appearance of the necessary number of auxiliary spin-tensor fields with lower generalized spins. We have shown that this purpose can be achieved with the help of an additional Fock space, by constructing an additive extension of a symmetry subsuperalgebra which consists of the subsystem of second-class constraints alone and of the generators of the Cartan subalgebra, which form an invertible even operator supermatrix composed of supercommutators of the second-class constraints.

We have realized the Verma module construction [47] in order to obtain an auxiliary representation in Fock space for the above superalgebra with second-class constraints. As a consequence, the converted Lie superalgebra of HS symmetry has the same algebraic relations as the initial superalgebra; however, these relations are realized in an enlarged Fock space. The generators of the converted Cartan subalgebra contain linearly two auxiliary independent number parameters, whose choice provides the vanishing of these generators in the corresponding subspaces of the total Hilbert space extended by the ghost operators in accordance with the minimal BFV-BRST construction for the converted HS symmetry superalgebra. Therefore, the above generators, enlarged by the ghost contributions up to the "particle number" operators in the total Hilbert space, covariantly determine Hilbert subspaces in each of which the converted symmetry superalgebra consists of the first-class constraints alone, labeled by the values of the above parameters, and constructed from the initial irreducible Poincare-group relations.

It is shown that the Lagrangian description corresponding to the BRST operator, which encodes the converted HS symmetry superalgebra, yields a consistent Lagrangian dynamics for fermionic fields of any generalized spin. The resulting Lagrangian description, realized concisely in terms of the total Fock space, presents a set of generating relations for the action and the sequence of gauge transformations for given fermionic HS fields with a sufficient set of auxiliary fields, and proves to be a reducible gauge theory with a finite number of reducibility stages, increasing with the value of generalized spin.

We have sketched a proof of the fact that the solutions of the Lagrangian equations of motion (5.24), (5.25), after a partial gauge-fixing, correspond to the BRST cohomology space with a vanishing ghost number, which is determined only by the relations that extract the fields of an irreducible Poincare-group representation with a given value of generalized spin.

As examples demonstrating the applicability of the general scheme, we have derived gauge-invariant Lagrangian formulations for the field of spin $(3/2, 1/2)$ and for the rank-2 antisymmetric spin-tensor in a manifest form in both massless and massive cases. In principle, the suggested algorithm permits one to derive manifest actions for any other half-integer spin fields characterized by two rows of the corresponding Young tableaux.

The basic results of the present work are given by relations (5.27), where the action for a field with an arbitrary generalized half-integer spin is constructed, as well as by relations (5.28)–(5.31), where the gauge transformations for the fields are presented, along with the sequence of reducible gauge transformations and gauge parameters.

JHEP10(2007)040

Concluding, we would like to discuss a number of additional points. First, the gauge-invariant description of massless and massive HS field theories with a mixed symmetry is an interesting starting point for a systematic construction of a Lagrangian formulation for HS interacting vertices with mixed-symmetry fermionic HS fields, including the case of the AdS space, in order to provide a description of the high-energy limit for open superstrings; see the arguments in favor of this suggestion in [39]. Second, the role of fermionic HS fields in the above limit of superstring theory in connection with the AdS/CFT correspondence signals the importance of extending the obtained results to the case of fermionic HS fields with a mixed symmetry in the AdS space. Thus, the present Lagrangian description takes a first step towards an interacting theory with mixed-symmetry fermionic HS fields, including the case of curved backgrounds, and then towards a covariant construction (following, e.g., the BV formalism) of the generating functionals of Green's functions, including the quantum effective action; examples of such calculations can be found, e.g., in [49]. Third, we estimate an extension of the obtained results to the case of arbitrary fermionic HS fields with any number of rows in the corresponding Young tableaux as a challenging technical problem. One of the possible approaches to this problem may rely on creating a computer algorithm which would permit one to obtain the HS symmetry superalgebra and calculate the action with the sequence of gauge transformations in an analytic component form for fermionic fields of a given generalized spin. Finally, the example of a field of spin $(3/2, 3/2)$ in section 7.2 has demonstrated a possibility of extracting a large number of auxiliary fields until the point when the component form of the action and gauge transformations can be derived in a manifest form. In our forthcoming work [50], we plan to realize this possibility, which should permit one to significantly reduce the spectrum of fields and gauge parameters in order to simplify the component structure of the basic results of the present work, however, with a possible appearance of additional off-shell constraints for the fields and gauge parameters.

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JHEP10(2007)040